

# Topological Membrane Solitons and Loop Orders of Membrane Scatterings in M-theory

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## Abstract

Two topological issues on membranes in M-theory are studied: (1) Soliton is an important subject in M-theory. Under the framework of obstruction theory with the help from framed links in  $S^3$ , we give a complete enumeration of topological membrane solitons in a string-admissible target-space of the form a product of Minkowskian space-times, tori, and K3-surfaces. Patching of these solitons and their topological charges are also defined and discussed. (2) Loop order of membrane scatterings is the basis for a perturbative M-theory. We explore this concept with emphases on its distinct features from pointlike and stringlike particles. For completeness, a light exposition on homologies of compact oriented 3-manifolds is given in the Appendix.

**Key Words:** Map-classes, obstruction theory, Heegaard splittings, framed links, Pontryagin-Thom construction, Cerf-Morse-Smale theory, Thurston's geometrization program.

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## 0. Introduction and outline.

### Introduction.

The string  $\beta$ -function at 1-loop level requires that the string target-space  $N$  be Ricci flat. Such string-admissible target-spaces include products of the Minkowskian space-time  $\mathbb{R}^{l+1} \times [0, 1]''$ , where  $[0, 1]$  is the quotient orbifold of  $S^1$  by an orientation-reversing diffeomorphism,  $n$ -dimensional tori  $\mathbb{T}^n$ , K3-surfaces, and Calabi-Yau 3-manifolds. Depending on the fermionic fields and supersymmetries involved in the theory, there are several types of strings (cf. [G-S-W]). Recently, the progress in our understanding of dualities indicates that these different types of strings may be viewed miraculously as "boundary values" or different "limits" of a master (or mother) theory, *M-theory* at an 11-dimensional string-admissible target-space. (Readers are referred to, for example, the physics literatures listed in Reference, which motivate and influence this paper.)

Exactly what M-theory is remains a mystery. Besides its relation to strings, it has 11-dimensional supergravity as a low energy limit. The theory contains not only membranes but also other higher dimensional  $p$ -branes moving around in a string-admissible target-space. Due to our limited knowledge of higher dimensional manifolds and general features of Calabi-Yau 3-manifolds as CW-complexes, we confine ourselves in this paper only to membranes moving in a string-admissible target-space  $N$  of the form

$$N = (\mathbb{E}^{l+1} \times [0, 1]') \times \mathbb{T}^n \times \text{K3-surface};$$

and consider two topological issues in M-theory: (1) topological membrane solitons in  $N$ , and (2) loop orders of membrane scatterings.

Just as the non-vanishness of homotopy groups provides the topological reason some solitons in gauge theory appear, the fact that there are non-trivial homotopy classes of maps from a membrane world-volume  $M^3$  into  $N$  can also provide the topological reason for the appearance of some membrane solitons in M-theory. Each homotopy class may represent a family of physical solitons that can be continuously deformed into each other and thus shall be regarded as a *topological soliton*. The set  $[M^3, N]$  of all of them should then play the same role to membrane soliton problems in M-theory as homotopy groups to soliton problems in gauge theory (cf. [Co]). Under the framework of obstruction theory with the help from framed links in  $S^3$ , we give a complete enumeration of the classes in  $[M^3, N]$ . Their patching and charges are also defined and discussed.

Next we consider another topological issue, the loop order of membrane scatterings. In string theory, the loop order of a world-sheet that describes a family of string scattering processes is a well-defined concept and it plays the role as an

expansion parameter for the perturbative string theory. We shall find that, once going beyond strings to higher dimensional extended objects, some essential features become very different and work remains to be done to fix this concept.

## Outline.

1. Essential mathematical backgrounds.
  2. Enumeration of topological membrane solitons in M-theory.
  3. Patching of membrane solitons and topological charges.
  4. Loop orders of membrane scatterings.
- Appendix: A light exposition on homologies of 3-manifolds.

*"I have merely transmitted what was taught to me . . . . ."*

————— Confucius, 551? – 479 B.C.

## 1 Essential mathematical backgrounds.

For the convenience of physicists and the introduction of notations, we sketch in this section some mathematical preliminaries needed for this paper. Some are given only in key words and more details are referred to the literatures.

• **Algebraic and differential topology.** ([B-T], [D-F-N], [Mi3], [Sp2], and [Vi].) CW-complexes  $X$ ; their homologies  $H_*(X; \mathbb{Z})$  and cohomologies  $H^*(X; \mathbb{Z})$ ; homotopy groups  $\pi_*(X, \cdot)$ ; Eilenberg-MacLane spaces  $K(\pi, n)$  as classifying spaces for cohomologies; Künneth formula; the universal coefficient theorem; homotopy classes of maps between two complexes; Hopf-Whitney classification theorem [Wh]; de Rham cohomology  $H_{DR}^*(N)$  for smooth manifold  $N$ ; relative de Rham cohomology  $H_{DR}^*(N, N_0)$ , where  $N_0$  is a submanifold of  $N$ ; Sard's theorem; Pontryagin-Thom construction.

When the base-point does not play roles in the discussion, we shall frequently omit it and denote the homotopy groups of  $X$  simply by  $\pi_*(X)$ .

• **Heegaard splitting.** ([He], [Ja], [Si] and [Sti].) A *handlebody*  $H_g$  of *genus*  $g$  is a 3-ball  $B^3$  with  $2g$  disjoint 2-discs in its boundary glued in pairs by an orientation-reversing homeomorphism. It has boundary a surface  $\Sigma_g$  of genus  $g$ . Every closed oriented 3-manifold  $M^3$  can be obtained by gluing two handlebodies of the same genus along their boundary. Such decomposition is called a *Heegaard splitting* of  $M^3$ . The data of the splitting can be coded by  $\Sigma_g$  together with two systems of

simple loops  $\{C_1, \dots, C_g\}$  and  $\{C'_1, \dots, C'_g\}$  with each system cutting  $\Sigma_g$  into a 2-sphere with  $g$  pairs of holes. From a Heegaard splitting, one has a special CW-complex structure of  $M^3$  constructed by first removing from  $\Sigma_g$  a 2-disc disjoint with all the loops to obtain a surface-with-one-hole  $\Sigma_g^\circ$ , next attaching a 2-cell  $e_i^2$  (and resp.  $e_i^{2'}$ ) along  $C_i$  (resp.  $C'_i$ ) to obtain a 2-complex  $X^2$ , and finally attaching a 3-cell  $e^3$  to  $X^2$  as indicated in FIGURE 1.1. We shall call this a *CW-complex structure of  $M^3$  associated to a Heegaard splitting*. The structure  $M^3 = X^2 \cup_h e^3$ , where  $h$  is the attaching map, shall be important to us later. Up to homotopy, we shall assume that  $h$  is an immersion on the complement of a 1-dimensional subset in  $\partial e^3$  (cf. FIGURES 1.1 and 2.1).

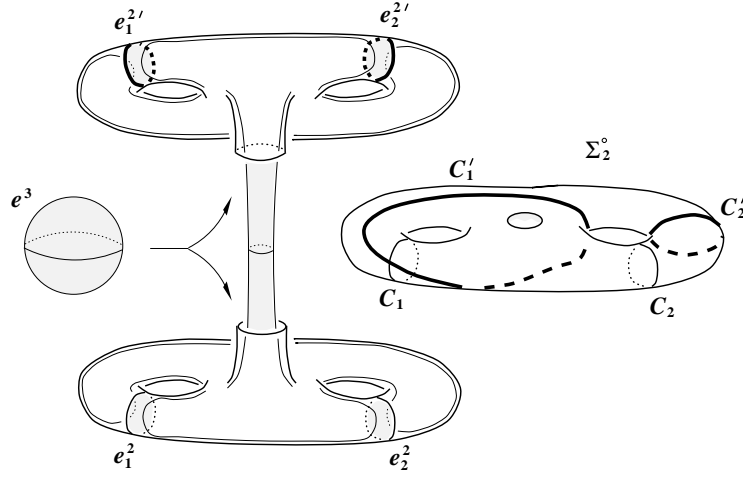


FIGURE 1.1. A CW-complex structure of  $M^3$  associated to a Heegaard splitting. This is an example of  $g = 2$ . For clarity, we duplicate the  $\Sigma_g^\circ$ -part of  $X^2$  in the picture, with one having  $e_i^2$  attached and the other having  $e_i^{2'}$  attached.

For this complex, recall the exact sequence

$$H_3(M^3, X^2; \mathbb{Z}) \xrightarrow{\partial} H_2(X^2; \mathbb{Z}) \longrightarrow H_2(M^3; \mathbb{Z}) \longrightarrow 0.$$

Since  $H_3(M^3, X^2; \mathbb{Z})$  is generated by  $e^3$  with  $\partial e^3 \sim 0$  in  $H_2(X^2; \mathbb{Z})$ , the inclusion map from  $X^2$  into  $M^3$  induces an isomorphism from  $H_2(X^2; \mathbb{Z})$  onto  $H_2(M^3; \mathbb{Z})$ . (Consequently,  $H_2(M^3; \mathbb{Z})$  is torsion free.)

Suppose that  $M^3$  is the connected sum  $M_1^3 \sharp M_2^3$  of two closed orientable 3-manifolds,  $M_1^3$ ,  $M_2^3$ . Let  $M_i^3 = X_i^2 \cup_{h_i} e_i^3$ ,  $i = 1, 2$ , from some Heegaard splittings. One can get a compatible complex structure  $M^3 = X^2 \cup_h e^3$  by first band-connected-summing  $X_1^2$ ,  $X_2^2$  along  $\partial \Sigma_{g_1}^\circ$  and  $\partial \Sigma_{g_2}^\circ$  to obtain a 2-complex  $X^2$ , and

then constructing  $(e^3, h)$  by amalgamating  $(e_1^3, h_1)$  and  $(e_2^3, h_2)$  together, following the pasting when connected-summing  $M_1^3$  and  $M_2^3$  (FIGURE 1.2, cf. [Ja]). Recall that  $H_i(M^3; \mathbb{Z}) = H_i(M_1^3; \mathbb{Z}) \oplus H_i(M_2^3; \mathbb{Z})$ , for  $i = 1, 2$ .

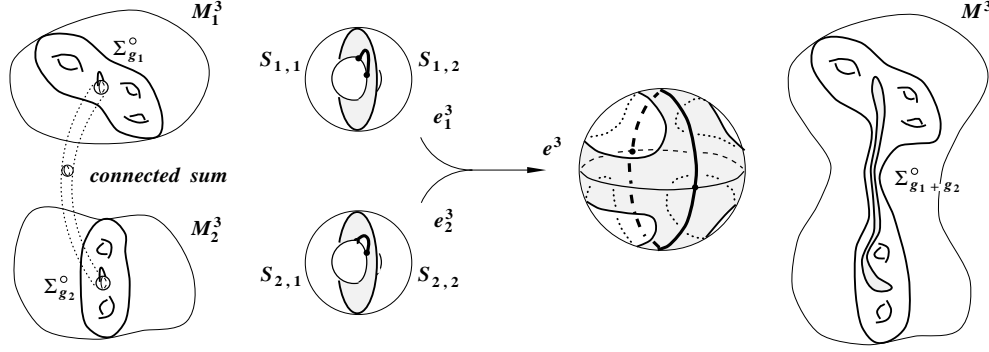


FIGURE 1.2. Connected-summing  $M_1^3$  and  $M_2^3$  along an embedded  $S^2$  in  $M_1^3$  that bounds a 3-ball  $B^3$  which intersects  $\Sigma_{g_1}^\circ$  with a half-2-disc along  $\partial\Sigma_{g_1}^\circ$  and a similar  $S^2$  in  $M_2^3$ . The relation between  $e_1^3$ ,  $e_2^3$  (with part of their boundary pasted along  $\partial\Sigma_{g_1}^\circ$  and  $\partial\Sigma_{g_2}^\circ$  respectively) and  $e^3$  are indicated. Each of the thickened-half-shells  $S_{1,1}$ ,  $S_{1,2}$ ,  $S_{2,1}$ , and  $S_{2,2}$  forms a quarter of  $e^3$  after pasting their inner boundary following the operation of connected sum. The unpasted outer boundaries become the four spots on  $\partial e^3$ . They are to be attached to  $\Sigma_{g_1+g_2}^\circ$  following their memories of the old attaching maps from  $M_1^3$  and  $M_2^3$ . Note that, after being pasted, the shaded part of  $\partial e^3$  forms a collar of  $\partial\Sigma_{g_1+g_2}^\circ$  in  $\Sigma_{g_1+g_2}^\circ$ .

• **Bouquets and framed links.** ([D-K], [Hi], [Ki], and [Mi3].) A *bouquet of 2-spheres*, denoted by  $\vee_r S^2$ , is a union of 2-spheres with a single common point (the *vertex*  $*$ ). Its homotopy groups have been studied by Hilton in [Hi]. We shall need the first three:

- (1)  $\pi_1(\vee_r S^2) = 0$ .
- (2)  $\pi_2(\vee_r S^2) = \oplus_r \mathbb{Z}$  is generated by the component  $S^2$ 's of the bouquet.
- (3)

$$\pi_3(\vee_r S^2) = (\oplus_r \mathbb{Z}) \oplus (\oplus_{\binom{r}{2}} \mathbb{Z}) = \oplus_{\frac{r(r+1)}{2}} \mathbb{Z}$$

is generated by two collections of elements: The generators  $[i]$ ,  $i = 1, \dots, r$ , of the first summand correspond to maps from  $S^3$  to some component  $S^2$  that give the Hopf fibration of  $S^3$ . As an attaching map of

a 4-cell, each gives a  $\mathbb{CP}^2$ . The generators  $[i, j]$ ,  $1 \leq i < j \leq r$ , of the second summand correspond to maps from  $S^3$  to some sub-bouquet  $S^2 \vee S^2$ , each of which, as an attaching map of a 4-cell, gives an  $S^2 \times S^2$ .

There is a geometric interpretation for  $\pi_3(\vee_r S^2)$ . Given a map  $f$  from an oriented  $S^3$  to an oriented  $\vee_r S^2$ . One may assume that  $f$  is smooth on the open subset of  $S^3$  that is mapped to the complement of the vertex  $*$ . By Sard's theorem, there exist regular values  $y_1, \dots, y_r$  of  $f$ , one on each oriented  $S^2$  in  $\vee_r S^2$ . The preimage  $f^{-1}(y_i)$  is an oriented framed link in  $S^3$ . For example, the framed links and their linking matrix associated to the generators  $[i]$  and  $[i, j]$  are indicated in FIGURE 1.3. Two maps  $f_1$  and  $f_2$  from  $S^3$  to  $\vee_r S^2$  are homotopic if and only if their associated

$$\begin{array}{ccc}
 \begin{array}{c} \text{Diagram of a single oriented framed link } [i] \\ \text{with a band indicating framing } i \end{array} & = & i \begin{array}{c} i \\ \mathbf{1} \end{array} \\
 [i] & & [i, j]
 \end{array}
 \qquad
 \begin{array}{ccc}
 \begin{array}{c} \text{Diagram of a linked oriented framed link } [i, j] \\ \text{with two bands indicating framings } i \text{ and } j \end{array} & = & \begin{array}{c} i \quad j \\ j \quad \mathbf{1} \end{array} \\
 [i, j] & & 
 \end{array}$$

FIGURE 1.3. The oriented framed link and the linking matrix associated to the generators  $[i]$  and  $[i, j]$  of  $\pi_3(\vee_r S^2)$ . The framing is indicated by a band along each component knot; and all the missing entries of the matrix are zero.

collections of framed links have the same linking matrix. On the other hand, any symmetric  $r \times r$  integral matrix can be realized as the linking matrix associated to some  $f$  by the Pontryagin-Thom construction. Thus one has a natural isomorphism between  $\pi_3(\vee_r S^2)$  and the additive group of symmetric  $r \times r$  integral matrices. Explicitly,

$$\sum_i c_i [i] + \sum_{i < j} c_{ij} [i, j] \quad \longleftrightarrow \quad \begin{pmatrix} c_1 & c_{12} & \cdots & c_{1r} \\ c_{12} & c_2 & \cdots & c_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1r} & c_{2r} & \cdots & c_r \end{pmatrix}.$$

• **Four-manifolds.** ([D-K], [Mi1], [Ki], and [Whi].) A compact simply-connected orientable 4-manifold is homotopic to a bouquet of 2-spheres attached with a 4-ball. In notation,  $M^4 = (\vee_r S^2) \cup_h e^4$ , where  $h$  is an attaching map. With respect to the basis consisting of the component  $S^2$ 's of the bouquet, the intersection form of  $H_2(M^4; \mathbb{Z})$  coincides with the linking matrix of the oriented framed links in  $S^3$  associated to  $h$ .

• **K3-surfaces.** ([D-K], [Ki], [Mi1], and [Sp1].) K3-surfaces are compact complex 2-manifolds with vanishing first Chern class. These Calabi-Yau 2-folds are all diffeomorphic to each other, whose topology can be constructed as follows. Observe that the  $\{\pm 1\}$ -action on a complex 4-torus  $\mathbb{T}^4$  by multiplication has  $2^4 = 16$  fixed points. This action extends to one on the blow-up  $\mathbb{T}^4 \# 16 \mathbb{CP}^2$  of  $\mathbb{T}^4$  at these fixed points. The latter action leaves the exceptional divisor -  $16 \mathbb{CP}^1$ 's - fixed. The quotient is then an oriented 4-manifold K3 that supports all K3-surfaces. Being simply-connected, its oriented homotopy type is determined by its intersection form

$$3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2(-E_8)$$

on  $H_2(\mathrm{K3}; \mathbb{Z})$ , where  $E_8$  is the positive-definite even form of rank 8 given by the matrix: (Only non-zero entries are shown.)

$$\begin{pmatrix} 2 & & -1 & & & & & & \\ & 2 & & -1 & & & & & \\ -1 & & 2 & & -1 & & & & \\ & -1 & -1 & & 2 & & -1 & & \\ & & & -1 & 2 & & -1 & & \\ & & & & -1 & 2 & & -1 & \\ & & & & & -1 & 2 & & -1 \\ & & & & & & -1 & 2 & \end{pmatrix}.$$

Up to homotopy,  $K3$  is a bouquet of 22 oriented 2-spheres attached with a 4-ball  $e^4$ . In notation,

$$\text{K3} = (V_{22} S^2) \cup_{h_0} e^4$$

for some attaching map  $h_0$ .

From the previous Items and Hurewicz isomorphism, one has

$$\pi_2(\mathrm{K}3) = H_2(\mathrm{K}3; \mathbb{Z}) = \oplus_{22} \mathbb{Z}$$

and

$$\pi_3(\text{K3}) = \pi_3(\vee_{22} S^2) / [h_0] \sim 0 \quad ,$$

where  $[h_0]$  is the class in  $\pi_3(\vee_{22} S^2)$  represented by  $h_0$ . Explicitly,

$$[h_0] = [1, 2] + [3, 4] + [5, 6] - 2 \sum_{i=7}^{22} [i] \\ + [7, 9] + [8, 10] + [9, 10] + [10, 11] + [11, 12] + [12, 13] + [13, 14] \\ + [15, 17] + [16, 18] + [17, 18] + [18, 19] + [19, 20] + [20, 21] + [21, 22]$$

(FIGURE 1.4). Since the coefficient of, say,  $[1, 2]$ -component in the combination is 1, one can replace  $[1, 2]$  by  $[h_0]$  to form a new basis of  $\pi_3(\vee_{22} S^2)$  with the rest of  $[i]$ 's and  $[i, j]$ 's. Consequently, after modding out  $[h_0]$ , one has

$$\pi_3(K3) = \oplus_{252} \mathbb{Z}.$$

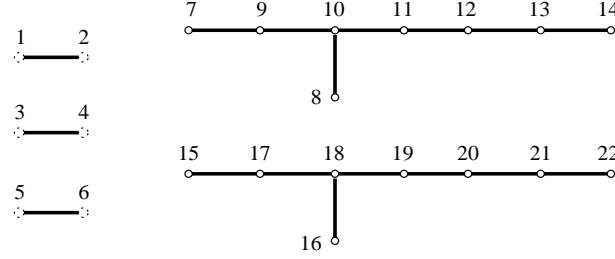


FIGURE 1.4. The intersection matrix of  $K3$ , coded in a Dynkin diagram, gives  $[h_0]$ . The dotted vertex indicates self-intersection number 0 and the undotted one  $-2$ . Each edge indicates intersection number 1.

• **Obstruction theory and enumeration of map-classes.** ([Hu], [Sp2], and [Wh].) Given two CW-complexes  $X$  and  $Y$ . A *map-class* from  $X$  to  $Y$  is a homotopy class of maps from  $X$  to  $Y$ . They form a set  $[X, Y]$  that depends only on the homotopy type of the source and target complexes. Let  $X^{(n)}$  be the  $n$ -skeleton of  $X$ . Two maps  $f_1, f_2$  from  $X$  to  $Y$  are said to be  $n$ -homotopic if the restrictions  $f_1|_{X^{(n)}}$ ,  $f_2|_{X^{(n)}}$  are homotopic. This is a topological concept, independent of what CW-complex structures are used for  $X$  and  $Y$  in the definition. An  $n$ -homotopy class represented by  $f$  shall be denoted by  $[f]_n$  and the set of all such classes by  $[X, Y]_n$ .

Assume that  $Y$  is path- and simply-connected. Let  $w$  be an  $n$ -homotopy class of maps from  $X$  to  $Y$  and  $Map(X^{(n)}, Y)$  be the space of all maps from  $X^{(n)}$  to  $Y$  (with the compact-open topology). Fix an  $f_0$  in  $w$  and define  $J_w$  to be the image of the following homomorphism:

$$\begin{aligned} \pi_1(Map(X^{(n)}, Y), f_0|_{X^{(n)}}) &\longrightarrow H^{n+1}(X; \pi_{n+1}(Y)) \\ [h_t] &\longmapsto \delta^{n+1}(f_0, f_0; h_t) \end{aligned} ,$$

where  $h_t$  is a homotopy of  $f_0|_{X^{(n)}}$  to itself, and  $\delta^{n+1}(f_0, f_0; h_t)$  is the *obstruction cohomology class* in  $H^{n+1}(X; \pi_{n+1}(Y))$  determined by  $f_0$  and  $h_t$ . (One may think of  $J_w$  as consists of all the "fake obstructions" for two  $n$ -homotopic  $f_1, f_2$  in  $w$  to be  $(n+1)$ -homotopic.) As the notation already indicates,  $J_w$  depends only on  $w$ . The



set of  $(n+1)$ -homotopy classes of maps in  $w$  can then be described by a subset  $A_{f_0}$  in the quotient  $H^{n+1}(X; \pi_{n+1}(Y))/J_w$ , which consists of all the  $J_w$ -orbit of obstruction classes  $\delta^{n+1}(f_0, f; h_t)$  in  $H^{n+1}(X; \pi_{n+1}(Y))$ , where  $f$  is in  $w$  and  $h_t$  is an  $n$ -homotopy between  $f_0$  and  $f$ . For different choices of  $f_0$  in  $w$ ,  $A_{f_0}$  differ by a translation in  $H^{n+1}(X; \pi_{n+1}(Y))/J_w$ .

## 2 Enumeration of topological membrane solitons in M-theory.

Given a compact oriented 3-manifold  $M^3$  that describes a membrane world-volume. Let  $N$  be the string-admissible target-space in the product form

$$N = \mathbb{R}^{l+1} \times [0, 1]^{l''} \times \mathbb{T}^n \times \text{K3}.$$

As explained and defined in the Introduction, the set of topological membrane solitons supported by  $M^3$  is then the set  $[M^3, N]$  of map-classes from  $M^3$  to  $N$ . Since

$$[M^3, N] = [M^3, \mathbb{R}^{l+1} \times [0, 1]^{l''}] \times [M^3, \mathbb{T}^n] \times [M^3, \text{K3}],$$

we only need to understand  $[M^3, \cdot]$  for each component of the product.

*Remark 2.1.* When  $\partial M^3$  is non-empty,  $[M^3, N]$  enumerates topological membrane solitons, supported by  $M^3$ , with free boundary. Let  $\partial_0 M^3$  be the union of some components of  $\partial M^3$ . Then one may also consider the set  $[(M^3, \partial_0 M^3), (N, \partial N)]$  of map-classes from  $M^3$  to  $N$  with  $\partial_0 M^3$  mapped to  $\partial N$ . They correspond to membrane solitons with part of the boundary components confined in the boundary of  $N$ . For the  $N$  in our problem, there is a natural surjection

$$[(M^3, \partial_0 M^3), (N, \partial N)] \longrightarrow [M^3, N],$$

the preimage of which at each point is isomorphic to  $[\partial_0 M^3, S^{l''-1}]$ . Explicitly,

$$\begin{aligned} l' = 1 & \quad [\partial_0 M^3, \{0, 1\}] = 2^{\pi_0(\partial_0 M^3)}, \text{ where } \pi_0(\partial_0 M^3) \text{ is the set of components of } \partial_0 M^3. \\ l' = 2 & \quad [\partial_0 M^3, S^1] = H^1(\partial_0 M^3; \mathbb{Z}). \\ l' = 3 & \quad [\partial_0 M^3, S^2] = \oplus_{\pi_0(\partial_0 M^3)} \mathbb{Z}, \text{ following from the fact that the homotopy class of a map from a connected oriented closed surface to } S^2 \text{ is determined by its degree.} \\ l' \geq 4 & \quad [\partial_0 M^3, S^{l''-1}] \text{ is a singleton; and hence } [(M^3, \partial_0 M^3), (N, \partial N)] \text{ and } [M^3, N] \text{ are canonically isomorphic.} \end{aligned}$$

Consequently, one only needs to understand  $[M^3, N]$ ; and all other boundary-confined situations then follow.

### Minkowskian and toroidal target-spaces.

The Minkowskian space-time  $\mathbb{R}^{l+1} \times [0, 1]^{l'}$  is contractible and hence contributes no obstructions to deformations of maps into it.

The circle  $S^1$  is an Eilenberg-MacLane space  $K(\mathbb{Z}, 1)$ . Thus the map-classes from  $M^3$  into it are completely classified by  $H^1(M^3; \mathbb{Z})$ . Consequently,

$$[M^3, \mathbb{T}^n] = \prod_n [M^3, S^1] = \prod_n H^1(M^3; \mathbb{Z}) = (\oplus_n \mathbb{Z})^{b_1},$$

where  $b_1$  is the first Betti number of  $M^3$ .

### K3 target-spaces: (1) When $\partial M^3$ is non-empty.

Since in this case  $M^3$  is homotopic to a 2-complex  $X^2$  and K3 is simply-connected, by the Hopf-Whitney classification theorem the map-classes from  $M^3$  to K3 are completely classified by elements in  $H^2(X^2; \pi_2(\text{K3})) = H^2(M^3; \pi_2(\text{K3}))$ . By the universal coefficient theorem and the fact that, for any finite abelian group  $A$ , the group of extensions  $\text{Ext}(A, B)$  is isomorphic to  $A \otimes B$  for any abelian group  $B$ , one has

$$[M^3, \text{K3}] = H^2(M^3; \pi_2(\text{K3})) = (\oplus_{22} \mathbb{Z})^{b_2} \oplus \left( \oplus_{22} \text{Tor } H_1(M^3; \mathbb{Z}) \right),$$

where  $b_2$  is the second Betti number of  $M^3$  and  $\text{Tor } H_1(M^3; \mathbb{Z})$  is the torsion part of  $H_1(M^3; \mathbb{Z})$ .

### K3 target-spaces: (2) When $M^3$ is closed.

Obstruction theory explains how maps may not be deformed into each other and hence different map-classes can arise, but in general it alone is not powerful enough to enumerate all the map-classes. In the current case, however, it turns out that, while under the frame work of obstruction theory, one can relate the problem to framed links in  $S^3$  and hence resolves the difficulties.

(i) *Preparations.* Given a Heegaard splitting of  $M^3$  along a closed oriented surface  $\Sigma_g$  with two systems of characteristic loops  $\{C_1, \dots, C_g\}$  and  $\{C'_1, \dots, C'_g\}$ . Let  $M^3 = X^2 \cup_h e^3$  be a CW-complex structure of  $M^3$  associated to the splitting. The attaching map  $h$  from  $\partial e^3 = S^2$  to  $X^2$  induces a decomposition of  $S^2$ : first, the loop

$h^{-1}(\partial\Sigma_g^\circ)$  separates  $S^2$  into the union of two hemispheres  $D_-^2$  and  $D_+^2$ ; and then the connected components of  $h^{-1}(C_i)$ ,  $h^{-1}(C'_i)$ , and  $h^{-1}(X^2 - \cup_i C_i - \cup_i C'_i)$  gives a further decomposition of each hemisphere. We shall denote the 2-dimensional components in  $D_-^2$  (resp.  $D_+^2$ ) by  $\Omega_j$  (resp.  $\Omega'_j$ ) for those from the  $\Sigma_g^\circ$  part and  $e_{i,1}^2$ ,  $e_{i,2}^2$  (resp.  $e_{i,1}^{2'}$ ,  $e_{i,2}^{2'}$ ) for those from the  $e_i^2$  (resp.  $e_i^{2'}$ ) part. By construction, there is an orientation-reversing homeomorphism  $\rho$  from  $\Omega_j$  to  $\Omega'_j$  and from  $e_{i,1}^2$  (resp.  $e_{i,1}^{2'}$ ) to  $e_{i,2}^2$  (resp.  $e_{i,2}^{2'}$ ) by sending  $p$  to  $\hat{p}$  if  $p$  and  $\hat{p}$  are attached to the same point on  $X^2$ . As a map on  $S^2$ ,  $\rho$  is in general discontinuous. (FIGURES 1.1 and 2.1.)

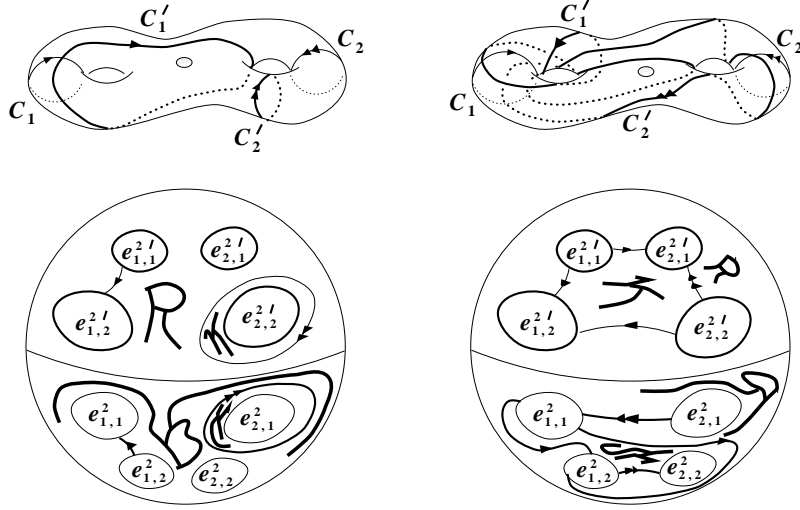


FIGURE 2.1. Two examples of the decomposition of  $S^2$  induced from Heegaard splittings. In each example, the map  $\rho$  is indicated by its effect on embedded letters K and R.

(ii) *2-homotopy classes.* Since K3 is simply-connected, the set of map-classes from  $X^2$  to K3 is enumerated by  $H^2(X^2; \pi_2(K3)) = H^2(M^3; \pi_2(K3))$ . On the other hand, every map  $f$  from  $X^2$  to K3 extends to  $M^3$  since the composition  $f \circ h$  from  $\partial e^3$  to K3 is always null-homotopic. Hence the set  $[M^3, K3]_2$  of 2-homotopy classes of maps from  $M^3$  to K3 is also enumerated by  $H^2(M^3; \pi_2(K3))$ .

(iii) *Map-classes.* Given a 2-homotopy class of maps from  $M^3$  to K3 represented by  $f$  and labelled by  $w$  in  $H^2(M^3; \pi_2(K3))$ . We shall first try to figure out the subgroup  $J_w$  in  $H^3(M^3; \pi_3(K3))$  and then the subset  $A_f$  in  $H^3(M^3; \pi_3(K3))/J_w$ .

Up to homotopy, we may assume that the image of  $f$  lies in  $\vee_{22} S^2$  with  $\Sigma_g^\circ$  all mapped to the vertex  $*$  and that  $f$  is smooth on the open subset of  $M^3$  that is mapped to the complement of  $*$ . Observe that there is a natural surjection from  $\pi_1(\text{Map}(X^2, \vee_{22} S^2), f|_{X^2})$  onto  $\pi_1(\text{Map}(X^2, K3), f|_{X^2})$  since every loop in

$Map(X^2, K3)$  at  $f|_{X^2}$  is homotopic to a loop in  $Map(X^2, \vee_{22} S^2)$  relative to the base point  $f|_{X^2}$ . Recall also that

$$H^3(M^3; \pi_3(\cdot)) = Hom(H_3(M^3; \mathbb{Z}), \pi_3(\cdot)) = \pi_3(\cdot)$$

since  $H_2(M^3; \mathbb{Z})$  is free. Consequently,  $J_w$  in  $H^3(M^3; \pi_3(K3))$  is the quotient of  $J_w$  in  $H^3(M^3; \pi_3(\vee_{22} S^2))$  by  $[h_0]$  and we only need to study the latter.

Consider now a map  $F : X^2 \times S^1 \rightarrow \vee_{22} S^2$  with  $F|_{X^2 \times \{0\}} = f|_{X^2}$ , which describes a loop in  $Map(X^2, \vee_{22} S^2)$  at the point  $f|_{X^2}$ . As a homotopy from  $f|_{X^2}$  to itself,  $F$  then defines an obstruction cohomology class  $\delta^3(f, f; F)$  in  $H^3(M^3; \pi_3(\vee_{22} S^2))$ .

**Lemma 2.2.** *Let  $W^2$  be the 2-subcomplex in  $X^2 \times S^1$  realized by*

$$(\Sigma_g^\circ \times \{0\}) \cup (\partial \Sigma_g^\circ \cup \cup_{i=1}^g C_i \cup \cup_{i=1}^g C'_i) \times S^1.$$

*Then  $F$  can be homotoped so that  $F$  sends  $W^2$  to the vertex  $*$  of  $\vee_{22} S^2$ , in addition to the requirement that  $F|_{X^2 \times \{0\}} = f|_{X^2}$ .*

Indeed  $F$  can be homotoped further so that it is smooth on the open subset of  $X^2 \times S^1$  that is mapped to the complement of  $*$ . And we shall call such  $F$  *nice*.

*Proof.* The second homology  $H_2(W^2; \mathbb{Z})$  of  $W^2$  is generated by the 2-cycles:  $\partial \Sigma_g^\circ \times S^1$  and  $C_i \times S^1$ ,  $C'_i \times S^1$ ,  $i = 1, \dots, g$ . Since each of  $\partial \Sigma^\circ$  and  $C_i$ ,  $C'_i$  bounds a 2-submanifold in  $X^2$ , each of the generating 2-cycles bounds a 3-submanifold in  $X^2 \times S^1$  and hence is homologous to 0 in  $H_2(X^2 \times S^1; \mathbb{Z})$ . Thus the induced map  $(F|_{W^2})_*$  from  $H_2(W^2; \mathbb{Z})$  to  $H_2(\vee_{22} S^2; \mathbb{Z})$  is a zero map. Since  $H_1(W^2; \mathbb{Z})$  is torsion free and  $\vee_{22} S^2$  is simply-connected, the Hopf-Whitney classification theorem then implies that  $F|_{W^2}$  is homotopic to the constant map that takes  $W^2$  to  $*$ . By the homotopy extension property of polyhedral pairs, one can extend this homotopy first to  $(\{e_i^2, e_i^{2'}\}_{i=1}^g \times \{0\}) \times [0, 1]$  via  $f|_{e_i^2}$  and  $f|_{e_i^{2'}}$ ; and then to the rest of  $(X^2 \times S^1) \times [0, 1]$ . This completes the proof.  $\square$

Fix a decomposition of  $S^3$

$$S^3 = e_-^3 \cup (S^2 \times [0, 1]) \cup e_+^3.$$

Up to homotopy, there is a unique (continuous) map  $\tau$  from  $S^3$  to  $M^3$  that satisfies: (1)  $\tau$  is a diffeomorphism on the interior of each of  $e_-^3$ ,  $e_+^3$  onto the interior of  $e^3$ , and (2)  $\tau|_{S^2 \times [0, 1]} = h|_{\partial e^3}$  via the projection from  $S^2 \times [0, 1]$  to  $S^2$ . The class  $\delta^3(f, f; F)$  can then be realized as a map  $G_{(f, f; F)}$  from  $S^3$  to  $\vee_{22} S^2$ , defined by

$$G_{(f, f; F)} = \begin{cases} f \circ \tau & \text{on } e_-^3 \cup e_+^3 \\ F \circ (h \times Id) & \text{on } S^2 \times [0, 1] \end{cases},$$

where  $Id$  is the identity map of the interval  $[0, 1]$ .

Up to homotopy, one may assume that  $F$  is nice. By Sard's theorem there exist common regular values of  $f$  and  $F$ :  $y_i$ ,  $i = 1, \dots, 22$ , one on each oriented  $S^2$  in the bouquet. Since the preimage  $F^{-1}(y_i)$  is now an oriented framed 1-manifold, with possibly several components, contained in the interior of the submanifold

$$X^2 \times S^1 - W^2 = (X^2 - \partial \Sigma_g^\circ - \cup_{i=1}^g C_i - \cup_{i=1}^g C'_i) \times S^1 - \Sigma_g^\circ \times \{0\}$$

in  $X^2 \times S^1$ , the preimage  $G_{(f,f;F)}^{-1}(y_i)$  is then a well-defined oriented framed link  $L_i$  in  $S^3$ . The linking matrix  $(lk(L_i, L_j))_{ij}$  indicates then what class  $G_{(f,f;F)}$  represents in  $H^3(M^3; \pi_3(\vee_{22} S^2)) = \pi_3(\vee_{22} S^2)$ .

Let us now take a closer look at this collection of links in  $S^3$ . Recall the map  $\rho$  on  $S^2$ . It extends naturally to  $S^2 \times [0, 1]$  via the product structure. We shall denote this new map still by  $\rho$ . Then, due to niceness of  $F$  and the construction, the framed 1-manifolds  $L_i \cap (S^2 \times [0, 1])$ , without orientation, are invariant under  $\rho$  with their boundary lying in the interior of the 2-discs

$$(\{e_{i,1}^2, e_{i,2}^2\}_{i=1}^g \times \{0, 1\}) \cup (\{e_{i,1}^{2'}, e_{i,2}^{2'}\}_{i=1}^g \times \{0, 1\}).$$

This renders a component knot  $K$  of the links be of only two types:

Type (a):  $K$  lies in the interior of  $\Omega \times [0, 1]$ , where  $\Omega$  is some  $\Omega_j$  or  $\Omega'_j$ .

Type (b):  $K$  lies in the interior of the handlebody of genus  $4g - 1$ ,

$$e_-^3 \cup (\{e_{i,1}^2, e_{i,2}^2, e_{i,1}^{2'}, e_{i,2}^{2'}\}_{i=1}^g \times [0, 1]) \cup e_+^3.$$

Consequently, one can *split* the links

$$L_i \longrightarrow L_i^- \cup L_i^0 \cup L_i^+$$

through a collection of disjoint segments in  $S^2 \times \{0, 1\}$  as indicated in FIGURE 2.2. Each of these segments as viewed from  $\Sigma_g^\circ$  is a dual loop to either some  $C_i$  in  $\{C_1, \dots, C_g\}$  or some  $C'_i$  in  $\{C'_1, \dots, C'_g\}$ . After being smoothed if necessary, the oriented framed links satisfy

$$lk(L_i, L_j) = lk(L_i^- \cup L_i^0 \cup L_i^+, L_j^- \cup L_j^0 \cup L_j^+).$$

After the splitting, each component knot  $K$  of Type (b) is either contained in  $e_-^3$ ,  $e_+^3$ , or a solid torus  $\Theta_i$ ,  $\Theta'_i$  (cf. FIGURE 2.3). The map  $\rho$  can be extended to each  $\Theta_i$  and  $\Theta'_i$ , on which it is an orientation-reversing homeomorphism. By construction,  $\rho$  leaves the links in a  $\Theta_i$  or  $\Theta'_i$  invariant with the framing preserved while the orientation reversed. The two collections of oriented framed links  $\{L_1^-, \dots, L_{22}^-\}$  and

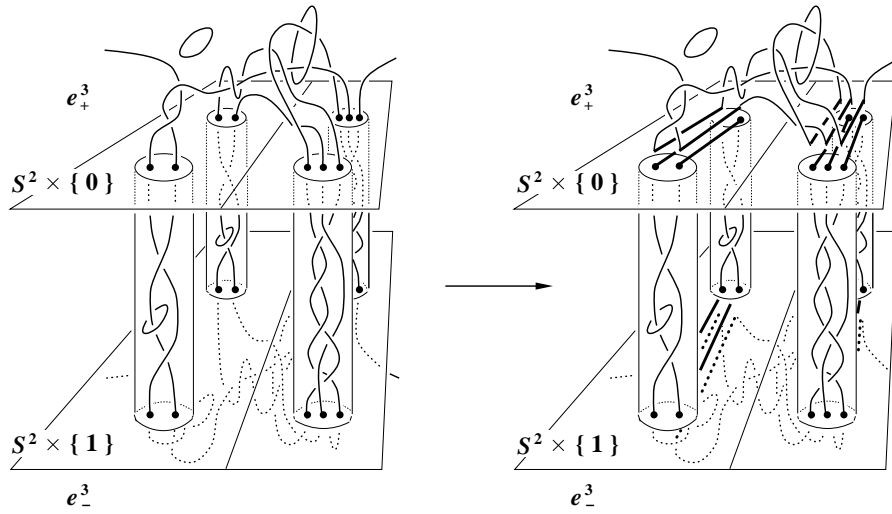


FIGURE 2.2. The splitting of  $L_i$  into  $L_i^- \cup L_i^0 \cup L_i^+$ . Up to homotopy, each component knot  $K$  of Type (b) is now contained in a solid torus  $\Theta_i$  or  $\Theta'_i$ . Those component knots of Type (a) are indifferent of such splittings and hence are omitted for the clarity of the picture. Notice that the  $\rho$  symmetry is maintained after the splitting and the collection of links in  $e_+^3$  and those in  $e_-^3$  are now the mirror-image of each other.

$\{L_1^+, \dots, L_{22}^+\}$  are contained in two separate 3-balls  $B_-^3$  and  $B_+^3$  that are disjoint from any of the  $L_j^0$ . The reflection between  $e_-^3$  and  $e_+^3$  with respect to the equator  $S^2 \times \{\frac{1}{2}\}$  in the decomposition of  $S^3$  induces an orientation-reversing homeomorphism between  $B_-^3$  and  $B_+^3$  that sends  $-$ -links and  $+$ -links to each other with the framing preserved while the orientation reversed. Together with the fact that the linking number of a pair of links remains the same if the orientation of both links are reversed and it differs exactly by a sign from that of their mirror-image pair, one concludes that

$$lk(L_i, L_j) = lk(L_i^-, L_j^-) + lk(L_i^0, L_j^0) + lk(L_i^+, L_j^+) = lk(L_i^0, L_j^0).$$

Note that this is expected from the fact that  $J_w$  depends only on  $w$ , not on any particular  $f$  that represents  $w$ .

Let  $K_1, K_2$  be component knots of some  $L_{i_0}^0, L_{j_0}^0$  and  $\Omega$  (resp.  $\Theta$ ) be some  $\Omega_i$  or  $\Omega'_i$  (resp.  $\Theta_i$  or  $\Theta'_i$ ). From the topological relations among regions  $\Omega \times [0, 1]$  and  $\Theta$  in  $S^3$ , there are only two situations that  $K_1, K_2$  may contribute to  $lk(L_{i_0}^0, L_{j_0}^0)$ :

- Case (1) Both  $K_1$  and  $K_2$  lie in a same  $\Omega \times [0, 1]$  or  $\Theta$ ,
- Case (2) Say,  $K_1$  lies in an  $\Omega \times [0, 1]$  and  $K_2$  lies in a  $\Theta$ .

In Case (1), the total contribution of such  $(K_1, K_2)$  to  $lk(L_{i_0}^0, L_{j_0}^0)$  vanishes due to the invariance of  $(L_{i_0}^0, L_{j_0}^0)$  under  $\rho$  up to the overall reversing of orientations. Consequently, the only contributions to the total linking number are from  $(K_1, K_2)$  in Case (2). In general this can be still complicated to analyze. However, there are simpler  $M^3$ , for which the task is straightforward.

Recall that an orientable 3-manifold  $M^3$  is called *prime* if any direct sum decomposition  $M^3 = M_1^3 \sharp M_2^3$  implies that either  $M_1^3$  or  $M_2^3$  is an  $S^3$ ; and is called *irreducible* if every embedded  $S^2$  bounds a 3-ball ([He]). The only prime, non-irreducible orientable 3-manifold is  $S^2 \times S^1$ . A 3-manifold is *reducible* if it is not prime.

Suppose that  $M^3$  is irreducible and that the Heegaard splitting in the discussion has the minimal genus. Consider  $K_1, K_2$  in Case (2).

**Lemma 2.3.** *Let  $M^3$  be an irreducible closed orientable 3-manifold with a Heegaard splitting of minimal genus*

$$M^3 = H_g \cup_{\Sigma_g} H'_g.$$

*Then every loop  $\gamma$  in a component  $\Xi$  of the complement of  $\{C_i, C'_i\}_{i=1}^g$  in  $\Sigma_g$  is null-homotopic in  $\Xi$ . In other words,  $\Xi$  is a disc.*

*Proof.* We only need to consider the case  $\gamma$  is a simple loop in  $\Xi$ . Since  $\gamma$  is contained in the complement of all  $C_i$  and  $C'_i$ , it is null-homotopic in both  $H_g$  and  $H'_g$ . By

Dehn's Lemma [He],  $\gamma$  bounds an embedded 2-disc  $\Delta^2$  in  $H_g$  and an embedded 2-disc  $\Delta^{2'}$  in  $H'_g$ . Together they give an embedded 2-sphere  $S^2$  in  $M^3$ . Since  $M^3$  is irreducible, this  $S^2$  bounds a 3-ball  $B^3$ . The intersection  $B^3 \cap \Sigma_g$  must then be a 2-disc in  $\Xi$  with boundary  $\gamma$  since otherwise the genus of the Heegaard splitting can be reduced, contradicting the minimal genus assumption. Therefore,  $\gamma$  is null-homotopic in  $\Xi$ . This concludes the lemma.  $\square$

Consequently, if  $\Omega$  does not contain  $h^{-1}(\partial\Sigma_g^\circ)$  in its boundary, then  $K_1$  does not link around  $K_2$  at all and hence  $lk(K_1, K_2) = 0$ . If, on the other hand,  $\Omega$  contains  $h^{-1}(\partial\Sigma_g^\circ)$  in its boundary, then, under homotopy, one may assume that  $K_1$  lies in a small neighborhood of  $h^{-1}(\partial\Sigma_g^\circ) \times [0, 1]$  and hence again does not link around  $K_2$  either. This shows that the linking matrix associated to  $G_{(f,f;F)}$  for any nice  $F$  is in fact the  $22 \times 22$  zero matrix; and  $G_{(f,f;F)}$  represents the zero class in  $\pi_3(\vee_{22} S^2)$ .

**Corollary 2.4.** *If  $M^3$  is irreducible, then  $J_w = 0$  for every  $w$  in  $H^2(M^3; \pi_2(K3))$ .*

For  $M^3 = M_1^3 \sharp M_2^3$ , there is a natural binary operation: ( $\cdot$  = either K3 or  $\vee_{22} S^2$ .)

$$\begin{aligned} + : H^3(M_1^3; \pi_3(\cdot)) \oplus H^3(M_2^3; \pi_3(\cdot)) &\longrightarrow H^3(M^3; \pi_3(\cdot)) \\ ([M_1^3] \mapsto \alpha_1), ([M_2^3] \mapsto \alpha_2) &\longmapsto ([M^3] \mapsto (\alpha_1 + \alpha_2)) \end{aligned}$$

which commutes with the identification of each of the  $H^3(\cdot; \pi_3(\cdot))$  with  $\pi_3(\cdot)$ . Under this operation, one has

**Lemma 2.5.** *Suppose that  $M^3$  can be decomposed into a connected sum  $M_1^3 \sharp M_2^3$ . Let  $w = w_1 + w_2$  following the decomposition*

$$H^2(M^3; \pi_2(K3)) = H^2(M_1^3; \pi_2(K3)) \oplus H^2(M_2^3; \pi_2(K3)).$$

*Then*

$$J_w = J_{w_1} + J_{w_2}.$$

*Proof.* Let  $M_i^3 = X_i^2 \cup_{h_i} e_i^3$ ,  $i = 1, 2$ , be a CW-complex structure of  $M_i^3$  induced from a Heegaard splitting. Let  $M^3 = X^3 \cup_h e^3$  be a compatible CW-complex structure for  $M^3$ . Then the relation between Heegaard splitting and connected sum, together with the discussion in this section, implies that the reduced oriented framed link  $L_i^0$  in  $S^3$  indeed separates into two parts  $L_{i,1}^0$  and  $L_{i,2}^0$ : the former corresponds to



the  $M_1^3$ -component and the latter to the  $M_2^3$ -component; and each is contained in a 3-ball (cf. FIGURE 1.2). Hence,

$$lk(L_i^0, L_j^0) = lk(L_{i,1}^0, L_{j,1}^0) + lk(L_{i,2}^0, L_{j,2}^0).$$

The lemma thus follows. □

Consequently,  $J_w$  vanishes for every  $w$  in  $H^2(M^3; \pi_2(K3))$  if  $M^3$  is the connected sum of a finite collection of irreducible orientable 3-manifolds.

If one decomposes a general  $M^3$  into

$$M^3 = M_0^3 \sharp_c (S^2 \times S^1),$$

where the prime decomposition of  $M_0^3$  contains no  $S^2 \times S^1$ . Recall from [Mi2] that both  $M_0^3$  and  $c$  are determined by  $M^3$ . Decompose  $H^2(M^3; \pi_2(K3))$  correspondingly as  $H^2(M_0^3; \pi_2(K3)) \oplus \oplus_c H^2(S^2 \times S^1; \pi_2(K3))$  and  $w$  as  $w_0 + w_1 + \cdots + w_c$ . Then

$$J_w = J_{w_1} + \cdots + J_{w_c}.$$

This somehow distinguishes the 3-manifold  $S^2 \times S^1$  in the current problem.

**Lemma 2.6.** *Let  $M^3 = S^2 \times S^1$  and  $w = (k_1, k_2, \dots, k_{22})$  be in  $H^2(S^2 \times S^1; \pi_2(K3)) = \oplus_{22} \mathbb{Z}$ . Then the abelian subgroup  $J_w$  in  $\pi_3(\vee_{22} S^2)$  is generated by*

$$\mathbf{u}_i = k_i[i] + \sum_{j < i} k_j[j, i] + \sum_{j > i} k_j[i, j] \quad \sim \quad \begin{pmatrix} & k_1 & & & \\ & \vdots & & & \\ k_1 & \cdots & k_i & \cdots & k_{22} \\ & \vdots & & & \\ & k_{22} & & & \end{pmatrix},$$

for  $i = 1, \dots, 22$ , where, for each  $i$ , the entries in the corresponding linking matrix are all zero except those in the  $i$ -th column or the  $i$ -th row. In particular,  $J_w$  is trivial for  $w = 0$  and is of rank 22 for  $w \neq 0$ .

*Proof.* A Heegaard splitting of  $S^2 \times S^1$  and its associate decomposition of  $S^2$  are indicated in FIGURE 2.3. Following the notations introduced in Sec. 1, note that  $H_2(X^2; \mathbb{Z}) \cong \mathbb{Z}$  is generated by  $e_1^2 - e_1^{2'}$ . One may assume then that  $f^{-1}(y_j)$  when restricted to  $X^2$  consists of  $k_j$  distinct points in the interior of  $e_1^2$  and none in the interior of  $e_1^{2'}$ . Subject to this, by the Pontryagin-Thom construction ([Mi3]), any oriented framed link as indicated in FIGURE 2.3 is realizable by some map  $f$  from  $S^2 \times S^1$  to  $\vee_{22} S^2$  in the given 2-homotopy class. The linking matrices and hence their

corresponding elements in  $\pi_3(\vee_{22} S^2)$  are exactly  $\mathbf{u}_i$ . Since only component knots in Case (2) contribute to the linking matrix, the combinations of integral copies of  $K_{(i)}$  as shown, which is part of the preimage  $f^{-1}(y_i)$ , exhaust all the possible linking matrices associated to maps in the 2-homotopy class  $w$ . This shows that  $\mathbf{u}_1, \dots, \mathbf{u}_{22}$  form a generating set for  $J_w$ . It is straightforward to check that they are linearly independent unless  $w = 0$ . This concludes the lemma.

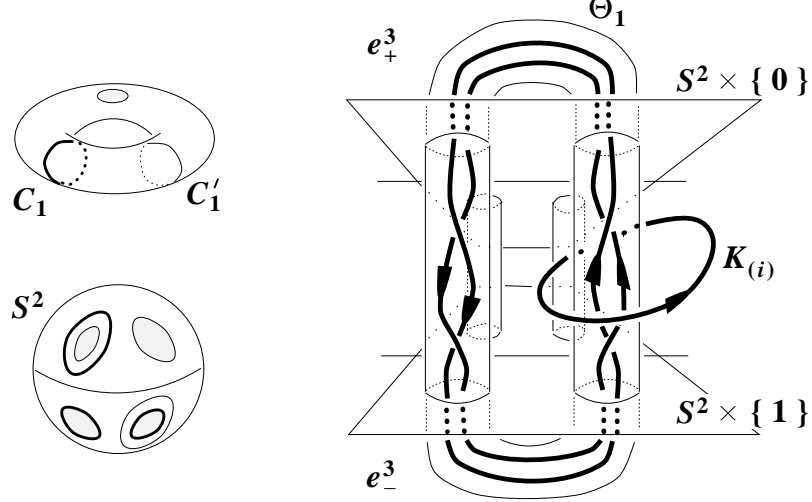


FIGURE 2.3. A Heegaard splitting of  $S^2 \times S^1$  by a torus and the corresponding decomposition of  $S^2$  are indicated. The collection of links in the solid torus  $\Theta_1$  depends on the 2-homotopy class  $w$ . The knot  $K_{(i)}$  in the preimage  $f^{-1}(y_i)$  gives  $\mathbf{u}_i$ . Together they generate  $J_w$ .

□

This concludes our discussion on  $J_w$ -part. As for  $A_f$ -part, notice that

**Lemma 2.7.** *Fix an  $f_0$  in  $w$ . Then every element in  $H^3(M^3; \pi_3(K3))$  can be represented by the obstruction class  $\delta^3(f_0, f; \cdot)$  for some  $f$  in  $w$ .*

*Proof.* Consider a trivial connected sum  $M^3 = M^3 \sharp S^3$  whose involved  $S^2$  is mapped to a point  $*$  in  $K3$  after a homotopy of  $f_0$ , and the 3-ball  $B^3$  it bounds is disjoint from  $X^2$ . Let  $g$  be a map from  $(B^3, S^2)$  to  $(K3, *)$ . Then the  $f$  defined on  $M^3$  by amalgamating  $f_0$  and  $g$ , following the trivial connected sum  $M^3 \sharp S^3$ , is 2-homotopic to  $f_0$  since  $f_0|_{X^2}$  and  $f|_{X^2}$  are identical. Now that  $g$  can be any class in  $\pi_3(K3)$ , one concludes the lemma.

□

*Remark 2.8.* Notice that, up to homotopy, every pair of 2-homotopic maps  $f_1, f_2$  from  $M^3$  to K3 are related as in the above proof.

Consequently, one has

$$[M^3, \text{K3}] = \bigcup_{[f_0]_2 \in H^2(M^3; \pi_2(\text{K3}))} A_{f_0} \simeq \bigcup_{w \in (\oplus_{22} \mathbb{Z})^{b_2} \oplus \oplus_{22} \text{Tor } H_1(M^3; \mathbb{Z})} \oplus_{253} \mathbb{Z} / [h_0] + J_w ,$$

where  $b_2$  is the second Betti number of  $M^3$  and  $\oplus_{253} \mathbb{Z} / ([h_0] + J_w)$  should be regarded only as a lattice (i.e. an integral affine space).

**Corollary 2.9.** *If the prime decomposition of  $M^3$  does not contain any  $S^2 \times S^1$  as a component (- note that this holds if and only if  $M^3$  contains no non-separating embedded  $S^2$  -), then*

$$\begin{aligned} [M^3, \text{K3}] &= H^2(M^3; \pi_2(\text{K3})) \times H^3(M^3; \pi_3(\text{K3})) \\ &\simeq \left( (\oplus_{22} \mathbb{Z})^{b_2} \oplus \oplus_{22} \text{Tor } H_1(M^3; \mathbb{Z}) \right) \times \oplus_{252} \mathbb{Z}. \end{aligned}$$

All together this concludes the enumeration of topological membrane solitons, represented by classes in  $[M^3, N]$ .

*Remark 2.10.* For  $N = \mathbb{R}^{3+1} \times [0, 1] \times \mathbb{T}^2 \times \text{K3}$  with complex Kähler  $\mathbb{T}^2 \times \text{K3}$ , two immediate problems follow: (1) *How can one characterize those topological membrane solitons that contain physical, say BPS-, solitons?* (2) *What is the moduli space of these physical solitons?* Both remain unsolved at the moment.

### 3 Patching of membrane solitons and topological charges.

In this section, we shall focus on the specific string-admissible target-space

$$N = \mathbb{R}^{3+1} \times [0, 1] \times \mathbb{T}^2 \times \text{K3},$$

which should play some role in the development of M-theory. The total dimension 11 has the origin from supergravity; and coincidentally the dimension 5 of the contractible part  $\mathbb{R}^{3+1} \times [0, 1]$  is the minimal dimension for a real vector space to allow every compact orientable 3-manifold to be embedded therein. Following from this, every class in  $[M^3, N]$  can be represented by an embedding.

### Patching of solitons and additivity of charges.

Let us consider the patchings first. Let  $(M_1^3, p_1)$  and  $(M_2^3, p_2)$  be two oriented closed 3-manifolds-with-base-point and let  $f_i$  be a map from  $M_i^3$  to  $N$  that represents a membrane soliton. If some  $M_i^3$  has non-empty boundary, then we require that  $p_i$  lies in the interior of  $M_i^3$ . Let  $\gamma$  be a path in  $N$  connecting  $f_1(p_1)$  and  $f_2(p_2)$ . One can then define a *patching of membrane solitons* in  $N$  by "puffing" the following joining of maps:

$$f_1 \cup \gamma \cup f_2 : M_1^3 \cup_{p_1 \sim 0} [0, 1] \cup_{1 \sim p_2} M_2^3 \longrightarrow N.$$

The result is an amalgamation  $f_1 \sharp_\gamma f_2$  from the connected sum  $M_1^3 \sharp M_2^3$  into  $N$ . (FIGURE 3.1.)

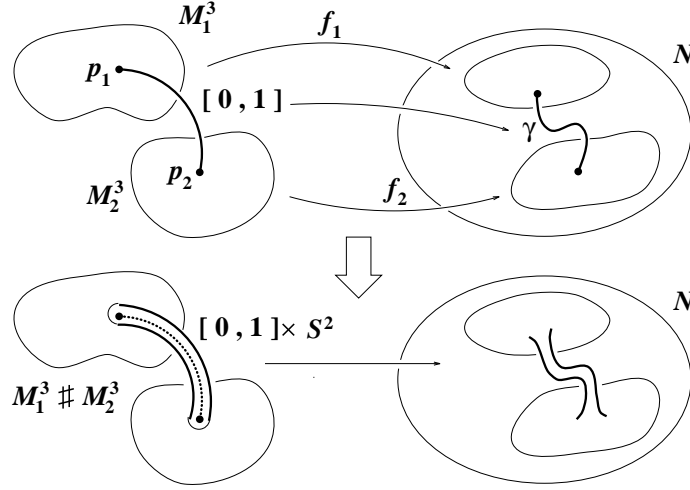


FIGURE 3.1. A map  $f_1 \sharp_\gamma f_2$  from  $M_1^3 \sharp M_2^3$  to  $N$  obtained by "puffing" along  $\gamma$ . This defines a patching of solitons.

In general, one would expect that different choices of  $p_1$ ,  $p_2$ , and  $\gamma$  may lead to non-homotopic maps from  $M_1^3 \sharp M_2^3$  to  $N$ . It turns out that this is not the case for our  $N$ . Let us look at the projection into the components  $\mathbb{T}^2$  and K3. Since  $M_1^3$ ,  $M_2^3$  are path-connected and  $\pi_1(M_1^3 \sharp M_2^3)$  is the free product  $\pi_1(M_1^3) * \pi_1(M_2^3)$ , the induced map  $(f_1 \sharp_\gamma f_2)_*$  is independent of the choices of  $p_1$ ,  $p_2$ , and  $\gamma$ . Hence, for the  $\mathbb{T}^2$ -component,  $f_1 \sharp_\gamma f_2$  gives a well-defined homotopy class  $[f_1 \sharp f_2]$  in  $[M_1^3 \sharp M_2^3, \mathbb{T}^2]$  since the latter is  $H^1(M_1^3 \sharp M_2^3; \pi_1(\mathbb{T}^2))$ , which is  $\text{Hom}(H_1(M_1^3 \sharp M_2^3; \mathbb{Z}), H_1(\mathbb{T}^2; \mathbb{Z}))$ . For the K3-component, since K3 is simply-connected, the same conclusion holds. Altogether, one has a well-defined class  $[f_1 \sharp f_2]$  in  $[M_1^3 \sharp M_2^3, N]$ . In other words, for our  $N$ , the patching thus defined is indeed an operation at the topological level.

Let us now consider the charges. For  $M^3$  closed, a map class  $[f]$  in  $[M^3, N]$  defines a class  $\langle f \rangle$  in  $H_3(N; \mathbb{Z})$ . Closed 3-forms on  $N$ , and hence classes in the de Rham cohomology  $H_{DR}^3(N)$ , can then be evaluated on  $\langle f \rangle$  and define various *topological charges* for the soliton  $f$ . By construction,

$$\langle f_1 \# f_2 \rangle = \langle f_1 \rangle + \langle f_2 \rangle.$$

And hence topological charges add under patchings.

For solitons with non-empty boundary, topological charges can also be defined for those whose boundary lies entirely in  $\partial N$ . Closed 3-forms on  $N$  that vanish at  $\partial N$ , and hence classes in the restricted relative de Rham cohomology  $H_{DR}^{(r)3}(N, \partial N)$ , can be evaluated on  $\langle f \rangle$  for such  $f$  and give the topological charges. Observe that

$$\left( H_{DR}^1([0, 1], \{0, 1\}) \otimes H_{DR}^2(\mathbb{T}^2 \times K3) \right) \oplus H_4(\mathbb{T}^2 \times K3; \mathbb{R}) \subset H_{DR}^{(r)3}(N, \partial N);$$

so there are actually a plenty source of charges for boundary-confined solitons in  $(N, \partial N)$ . The additivity property of charges under patchings still holds.

As a comparison, the patching of membrane solitons defined here generalizes the patching that occurs in gauge theory. In the latter case, homotopy groups and, hence, amalgamations of maps from connected-sums of spheres are involved ([Co]). The fact that topological charges add under patchings indicate that such generalization is natural for  $N$ .

Since topological charges depend only on  $\langle f \rangle$ , the information is contained entirely in the *tautological map* either from  $[(M^3, \partial M^3), (N, \partial N)]$  to  $H_3(N, \partial N; \mathbb{Z})$  if  $\partial M^3$  is non-empty, or from  $[M^3, N]$  to  $H_3(N; \mathbb{Z})$  if  $M^3$  is closed, that sends  $[f]$  to  $\langle f \rangle$ . We shall now study some details of this correspondence.

### The tautological map: (1) When $\partial M^3$ is nonempty.

Let  $\partial N = \mathbb{R}^{3+1} \times \{0, 1\} \times \mathbb{T}^2 \times K3 = \partial_0 N \cup \partial_1 N$ . From the contractibility of  $\mathbb{R}^{3+1}$  and  $[0, 1]$ , and the Poincaré and the Poincaré-Lefschetz dualities, one has

$$\begin{aligned} H_3(N, \partial N; \mathbb{Z}) &= H^4([0, 1] \times \mathbb{T}^2 \times K3; \mathbb{Z}) = H_2(\mathbb{T}^2 \times K3; \mathbb{Z}) \\ &= H_2(\mathbb{T}^2; \mathbb{Z}) \oplus H_2(K3; \mathbb{Z}) = H_2(\partial_0 N; \mathbb{Z}) = H_2(\partial_1 N; \mathbb{Z}). \end{aligned}$$

As a result,  $H_3(N, \partial N; \mathbb{Z})$  is generated by the embedded 3-manifolds-with-boundary  $[0, 1] \times \mathbb{T}^2$  and  $[0, 1] \times S_j^2$ ,  $j = 1, \dots, 22$ .

Let  $\partial M = \Sigma_{(1)}^2 \cup \dots \cup \Sigma_{(r)}^2$  and  $f : (M^3, \partial M^3) \rightarrow (N, \partial N)$ . If  $f(\partial M)$  is contained completely in, say,  $\partial_0 N$ , then  $f$  can be homotoped into  $\partial_0 N$ , thus  $\langle f \rangle = 0$  in  $H_3(N, \partial N; \mathbb{Z})$ . In general, up to a relabelling, suppose that  $f$  maps the first

$r_0$  components of  $\partial M$  into  $\partial_0 N$  and the rest into  $\partial_1 N$ . Then, under the above isomorphism,

$$\begin{aligned}\langle f \rangle \text{ in } H_3(N, \partial N; \mathbb{Z}) &= \langle f|_{\Sigma_{(r_0+1)}^2} \rangle + \cdots + \langle f|_{\Sigma_{(r)}^2} \rangle \text{ in } H_2(\partial_1 N; \mathbb{Z}) \\ &= -\langle f|_{\Sigma_{(1)}^2} \rangle - \cdots - \langle f|_{\Sigma_{(r_0)}^2} \rangle \text{ in } H_2(\partial_0 N; \mathbb{Z})\end{aligned}$$

since such  $f$  differs from one that takes  $\partial M$  all into  $\partial_0 N$  by a dragging of  $\Sigma_{(r_0+1)}^2, \dots, \Sigma_{(r)}^2$  from  $\partial_0 N$  to  $\partial_1 N$ . Therefore, if let  $pr_1, pr_2$  be the projection maps from  $N$  to  $\mathbb{T}^2$  and K3 respectively, then

$$\begin{aligned}\langle f \rangle &= \left( \langle (pr_1 \circ f)|_{\Sigma_{(r_0+1)}^2} \rangle + \cdots + \langle (pr_1 \circ f)|_{\Sigma_{(r)}^2} \rangle \right) \\ &\quad \oplus \left( \langle (pr_2 \circ f)|_{\Sigma_{(r_0+1)}^2} \rangle + \cdots + \langle (pr_2 \circ f)|_{\Sigma_{(r)}^2} \rangle \right).\end{aligned}$$

From this expression, one can obtain  $\langle f \rangle$  from  $[f]$  as follows.

Let  $[\mathbb{T}^2]$  be the fundamental class of  $\mathbb{T}^2$  in  $H_2(\mathbb{T}^2; \mathbb{Z})$ . Then the first summand in the above expression is the summation of  $\deg(pr_1 \circ f) \cdot [\mathbb{T}^2]$  from the involved component  $\Sigma^2$  of  $\partial M^3$  to  $\mathbb{T}^2$ . This can be computed, through Lemma 3.1 below, from  $i_* H_1(\Sigma^2; \mathbb{Z})$ , the image of  $H_1(\Sigma^2; \mathbb{Z})$  in  $H_1(M^3; \mathbb{Z})$  under the inclusion map  $i$  from  $\partial M^3$  to  $M^3$ , and any 1-cocycle of  $M^3$  with coefficient in  $\pi_1(\mathbb{T}^2)$  that represents  $[pr_1 \circ f]$ . Similarly, the second summand can be computed from  $i_*(\Sigma)$  in  $H_2(M^3; \mathbb{Z})$  and any 2-cocycle of  $M^3$  with coefficient in  $\pi_2(\text{K3})$  that represents  $[pr_2 \circ f]$ . Explicit formula depends on the detail of the inclusion map  $i$ .

**Lemma 3.1.** *Let  $f$  be a map from a closed oriented surface  $\Sigma$  to an oriented torus  $\mathbb{T}^2$ . Let  $(a_1, b_1, \dots, a_g, b_g)$  be a canonical basis for  $H_1(\Sigma; \mathbb{Z})$ . Suppose that, with respect to a basis for  $H_1(\mathbb{T}^2; \mathbb{Z})$ ,  $f_*(a_i) = (k_i, l_i)$  and  $f_*(b_i) = (m_i, n_i)$ , then the degree of  $f$  is determined by*

$$\deg f = \begin{vmatrix} k_1 & l_1 \\ m_1 & n_1 \end{vmatrix} + \cdots + \begin{vmatrix} k_g & l_g \\ m_g & n_g \end{vmatrix}.$$

*Proof.* We shall not distinguish  $a_i, b_i$  with their corresponding simple loops in  $\Sigma$ . Under homotopy of loops, one may assume that  $a_i, b_i$  are loops at a  $p_0$  in  $\Sigma$  and that the complement of the loops is a  $4g$ -gon  $\Delta^2$ , whose boundary is given by the edge-loop  $\Pi_{i=1}^g [a_i, b_i]$ . Let  $\mathbb{T}^2 = \mathbb{R}^2 / (\mathbb{Z} \oplus \mathbb{Z})$  with respect to the lifting of the given basis for  $\mathbb{T}^2$ . Then the restriction  $f|_{\Delta^2}$  can be lifted to a map  $\tilde{f}$  from  $\Delta^2$  to  $\mathbb{R}^2$ . With an appropriate introduction of coordinates and some abuse of notations, let

$dx, dy$  be the 1-forms on  $\mathbb{T}$  dual to the given basis. We shall denote their lifting to the covering  $\mathbb{R}^2$  also by  $dx$  and  $dy$ . Then

$$\deg f = \int_{\Sigma} f^*(dx \wedge dy) = \int_{\Delta^2} \tilde{f}^*(dx \wedge dy) = \frac{1}{2} \int_{\partial \Delta^2} \tilde{f}^*(x dy - y dx).$$

As indicated in FIGURE 3.2, the loop  $\tilde{f} \circ \partial \Delta^2$  sweeps out  $g$  curvy parallelograms, with one for each  $[a_i, b_i]$ . The formula then follows. This concludes the lemma.

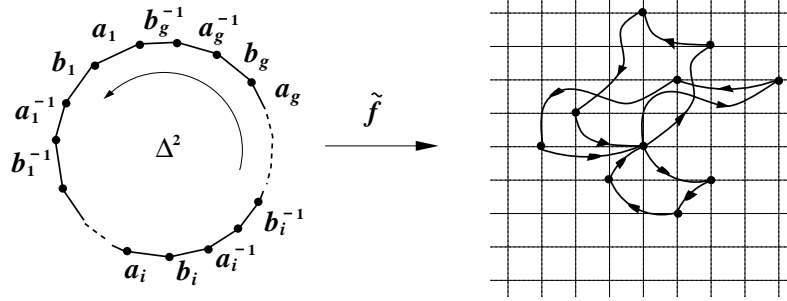


FIGURE 3.2. The image of  $\partial \Delta^2$  in the covering space  $\mathbb{R}^2$  of  $\mathbb{T}^2$  under  $\tilde{f}$  sweeps out  $g$  curvy parallelograms. Three of them are shown.

□

### The tautological map: (2) When $M^3$ is closed.

In the following discussion, we shall not distinguish the map  $f$  from  $M^3$  into  $N$  and its projection into  $\mathbb{T}^2 \times K3$ .

Recall from Remark 2.8 that up to homotopy any two representatives for a 2-homotopy class  $[f]_2$  from  $M^3$  to  $N$  differ by an amalgamation with a map  $g$  from  $S^3$  to  $N$ . Observe also that the Hurewicz homomorphism from  $\pi_3(\mathbb{T}^2 \times K3)$  into  $H_3(\mathbb{T}^2 \times K3; \mathbb{Z})$  is a zero map since  $\pi_3(\mathbb{T}^2 \times K3) = \pi_3(K3)$  and  $H_3(K3; \mathbb{Z}) = 0$ . Consequently,

$$\langle f + g \rangle = \langle f \rangle + \langle g \rangle = \langle f \rangle.$$

and one has

**Lemma 3.2.** *The homology class  $\langle f \rangle$  in  $H_3(\mathbb{T} \times K3; \mathbb{Z})$  depends only on the 2-homotopy class  $[f]_2$  of  $f$ .*

Let  $[M^3, N]_2$  be the set of 2-homotopy classes of maps from  $M^3$  to  $N$ . Recall from Sec. 2 that

$$\begin{aligned} [M^3, N]_2 &= H^1(M^3; \pi_1(\mathbb{T}^2)) \times H^2(M^3; \pi_2(K3)) \\ &= (\oplus_2 \mathbb{Z})^{b_1} \times \left( (\oplus_{22} \mathbb{Z})^{b_2} \oplus \oplus_{22} \text{Tor } H_1(M^3; \mathbb{Z}) \right) \end{aligned}$$

with respect to a basis  $(S_1^1, S_2^1)$  for  $\pi_1(\mathbb{T}^2) = H_1(\mathbb{T}^2; \mathbb{Z})$  and the basis  $(S_1^2, \dots, S_{22}^2)$  for  $\pi_2(K3) = H_2(K3; \mathbb{Z})$  from the component 2-spheres in  $K3 = (\vee_{22} S^2) \cup_{h_0} e^4$ . By Künneth formula,

$$H_3(N; \mathbb{Z}) = H_3(\mathbb{T}^2 \times K3; \mathbb{Z}) = H_1(\mathbb{T}^2; \mathbb{Z}) \otimes H_2(K3; \mathbb{Z})$$

is generated by the 44 classes  $\langle S_i^1 \times S_j^2 \rangle$ ,  $i = 1, 2$  and  $j = 1, \dots, 22$ . With respect to this tensor product structure and the basis, the intersection form for  $H_3(\mathbb{T}^2 \times K3; \mathbb{Z})$  can be expressed in the following block form

$$\begin{pmatrix} \text{O}_{22 \times 22} & 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2(-E_8) \\ 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2(-E_8) & \text{O}_{22 \times 22} \end{pmatrix},$$

where  $\text{O}_{22 \times 22}$  is the  $22 \times 22$  zero matrix. This matrix is non-degenerate, whose inverse is given by

$$\begin{pmatrix} \text{O}_{22 \times 22} & 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2(-E_8^{-1}) \\ 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2(-E_8^{-1}) & \text{O}_{22 \times 22} \end{pmatrix},$$

where

$$E_8^{-1} = \begin{pmatrix} 4 & 5 & 7 & 10 & 8 & 6 & 4 & 2 \\ 5 & 8 & 10 & 15 & 12 & 9 & 6 & 3 \\ 7 & 10 & 14 & 20 & 16 & 12 & 8 & 4 \\ 10 & 15 & 20 & 30 & 24 & 18 & 12 & 6 \\ 8 & 12 & 16 & 24 & 20 & 15 & 10 & 5 \\ 6 & 9 & 12 & 18 & 15 & 12 & 8 & 4 \\ 4 & 6 & 8 & 12 & 10 & 8 & 6 & 3 \\ 2 & 3 & 4 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}.$$

Hence one can identify  $\langle f \rangle$  by its intersection numbers with the 3-cycles in the basis.

To understand these numbers, let us fix a Heegaard splitting  $M^3 = H_g \cup_{\Sigma_g} H'_g$  and recall the associated complex  $M^3 = X^2 \cup_h e^3$ . Define a *characteristic system of solid tori* in  $M^3$  with respect to the splitting to be a collection of disjoint embedded



solid tori  $(S^1 \times D^2)_i$ ,  $(S^1 \times D^2)_{i'}$ ,  $i = 1, \dots, g$ , such that (1)  $(S^1 \times D^2)_i$  (resp.  $(S^1 \times D^2)_{i'}$ ) is contained in the interior of  $H_g$  (resp.  $H'_g$ ), and (2) the core  $S^1 \times \{0\}$  of  $(S^1 \times D^2)_i$  (resp.  $(S^1 \times D^2)_{i'}$ ) intersects  $e_i^2$  (resp.  $e_{i'}^{2'}$ ) exactly once and is disjoint from all other  $e_j^2$  (resp.  $e_j^{2'}$ ) (FIGURE 3.3). It follows that the collection of the core loops of the  $g$  tori in  $H_g$  form a generating set for  $H_1(M^3; \mathbb{Z})$ ; and similarly for those in  $H'_g$ .

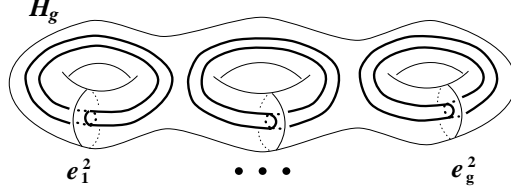


FIGURE 3.3. A characteristic system of solid tori in  $M^3$  with respect to a Heegaard splitting  $M^3 = H_g \cup_{\Sigma_g} H'_g$ . Only the part in  $H_g$  are shown.

Let  $pr_1$ ,  $pr_2$  be the projection map from  $\mathbb{T}^2 \times K3$  into  $\mathbb{T}^2$  and  $K3$  respectively. In the 2-homotopy class  $[f]_2$ , one can choose  $f$  such that the image of  $pr_2 \circ f$  lies in  $\vee_{22} S^2$  with the preimage  $(pr_2 \circ f)^{-1}(\vee_{22} S^2 - \{*\})$  a collection of disjoint embedded solid tori

$$\left( \bigcup_{i,j;r} (S^1 \times D^2)_{i,j;r} \right) \cup \left( \bigcup_{i',j;r'} (S^1 \times D^2)_{i',j;r'} \right)$$

in  $M^3$  that satisfy: (1) every  $(S^1 \times D^2)_{i,\cdot,\cdot}$  (resp.  $(S^1 \times D^2)_{i',\cdot,\cdot}$ ) is isotopic to  $(S^1 \times D^2)_i$  (resp.  $(S^1 \times D^2)_{i'}$ ); and (2) every  $(S^1 \times D^2)_{\cdot,j,\cdot}$  is mapped to  $S_j^2 - \{*\}$  under  $pr_2 \circ f$ . Let  $\alpha_i$  be the Poincaré dual of  $S_i^1$  in  $\mathbb{T}^2$ ,  $\beta_j$  be the Poincaré dual of  $S_j^2$  in  $K3$ . Then, with some abuse of notations, the intersection number of  $\langle f \rangle$  with  $\langle S_{i_0}^1 \times S_{j_0}^2 \rangle$  is given by

$$\begin{aligned} \langle f \rangle \cdot \langle S_{i_0}^1 \times S_{j_0}^2 \rangle &= \int_{\bigcup_{i,j;r} (S^1 \times D^2)_{i,j;r}} f^*(\alpha_{i_0} \wedge \beta_{j_0}) \\ &\quad + \int_{\bigcup_{i',j;r'} (S^1 \times D^2)_{i',j;r'}} f^*(\alpha_{i_0} \wedge \beta_{j_0}) \\ &= \sum_{i,j;r} \left( \int_{\gamma_{i,j;r}} \alpha_{i_0} \int_{D_{i,j;r}^2} \beta_{j_0} \right) + \sum_{i',j;r'} \left( \int_{\gamma_{i',j;r'}} \alpha_{i_0} \int_{D_{i',j;r'}^2} \beta_{j_0} \right), \end{aligned}$$

where  $\gamma_{i,j;r}$  is the core  $S^1 \times \{0\}$  and  $D_{i,j;r}^2$  the slice  $\{0\} \times D^2$  in  $(S^1 \times D^2)_{i,j;r}$  and similarly for  $\gamma_{i',j;r'}$  and  $D_{i',j;r'}^2$ . But this integral is exactly the intersection number of the class

$$\Psi_f = \sum_{i,j;r} \langle (g^{(1)} \times g^{(2)})_{i,j;r} \rangle + \sum_{i',j;r'} \langle (g^{(1)} \times g^{(2)})_{i',j;r'} \rangle$$

with  $\langle S_{i_0}^1 \times S_{j_0}^2 \rangle$ , where  $(g^{(1)} \times g^{(2)})_{i,j;r}$  is the map from  $S^1 \times S^2$  to  $\mathbb{T}^2 \times \text{K3}$  given by

$$(g^{(1)} \times g^{(2)})_{i,j;r} = (pr_1 \circ f)|_{\gamma_{i,j;r}} \times (pr_2 \circ f)|_{D_{i,j;r}^2};$$

and similarly for  $(g^{(1)} \times g^{(2)})_{i',j';r'}$ . This shows that indeed

$$\langle f \rangle = \Psi_f.$$

Explicitly, let  $\widehat{C}_i$  (resp.  $\widehat{C}'_i$ ) be a loop on  $\Sigma_g$  homotopic to the core of  $(S^1 \times D^2)_i$  (resp.  $(S^1 \times D^2)_{i'}$ ) in  $H_g$  (resp.  $H'_g$ ). If  $[pr_1 \circ f]$  is represented by a 1-cocycle with coefficients in  $\pi_1(\mathbb{T}^2)$

$$\begin{aligned} \eta_f : \{ \widehat{C}_1, \dots, \widehat{C}_g, \widehat{C}'_1, \dots, \widehat{C}'_g \} &\longrightarrow \pi_1(\mathbb{T}^2) = H_1(\mathbb{T}^2; \mathbb{Z}) \\ \widehat{C}_s &\longmapsto c_{s,1} \langle S_1^1 \rangle + c_{s,2} \langle S_2^1 \rangle \\ \widehat{C}'_s &\longmapsto c'_{s,1} \langle S_1^1 \rangle + c'_{s,2} \langle S_2^1 \rangle \quad , \end{aligned}$$

and  $[pr_2 \circ f]_2$  by a 2-cocycle with coefficients in  $\pi_2(\text{K3})$

$$\begin{aligned} \xi_f : \{ e_1^2, \dots, e_g^2, e_1^{2'}, \dots, e_g^{2'} \} &\longrightarrow \pi_2(\text{K3}) = H_2(\text{K3}; \mathbb{Z}) \\ e_s^2 &\longmapsto d_{s,1} \langle S_1^2 \rangle + \dots + d_{s,22} \langle S_{22}^2 \rangle \\ e_s^{2'} &\longmapsto d'_{s,1} \langle S_1^2 \rangle + \dots + d'_{s,22} \langle S_{22}^2 \rangle \quad , \end{aligned}$$

then  $\langle f \rangle$  is the class

$$\begin{aligned} \langle f \rangle &= \sum_{s=1}^g \eta_f(\widehat{C}_s) \otimes \xi_f(e_s^2) + \sum_{s'=1}^g \eta_f(\widehat{C}'_{s'}) \otimes \xi_f(e_{s'}^{2'}) \\ &= \sum_{\substack{i=1,2 \\ j=1, \dots, 22}} (c_{s,i} d_{s,j} + c'_{s,i} d'_{s,j}) \langle S_i^1 \times S_j^2 \rangle . \end{aligned}$$

in  $H_3(N; \mathbb{Z})$ .

## 4 Loop orders of membrane scatterings.

The loop order of a scattering process in quantum field and string theory indicates the complexity of that process and serves as an expansion parameter for the related perturbative theory. In this last section, we discuss what happens for higher dimensional extended objects, particularly membranes.

There are two aspects of membrane scatterings:

- (1) *The Hamiltonian aspect:* Evolution of membranes and their interactions at various instances are emphasized. Hence the Cerf-Morse-Smale theory on 3-manifolds plays the key.
- (2) *The Lagrangian aspect:* The membrane world-volume is treated as a whole. Hence the topology of 3-manifolds plays the key.

Let us thus take a look at both aspects. Physicists are referred to [Ce], [D-F-N], [He], [Ja], [M-B], [M-T], [Sco], [Sta], [Th1] and [Th2] for a survey and details of miscellaneous mathematics used in this section.

### The Hamiltonian aspect.

Let us begin with a list of essential objects that describe a membrane scattering and their mathematical equivalent in Cerf-Morse-Smale theory.

<i>Membrane scattering:</i>	<i>Cerf-Morse-Smale theory:</i>
<ul style="list-style-type: none"> <li>• <i>Time-function</i> <math>\tau</math> on the membrane world-volume <math>M^3</math> induced from the space-time where membranes are moving around.</li> <li>• Equal-time <i>slicing</i> of <math>M^3</math>.</li> <li>• <i>Joining</i> of two membranes <math>\Sigma_1^2, \Sigma_2^2</math>.</li> <li>• <i>Splitting</i> of a membrane <math>\Sigma^2</math>.</li> <li>• <i>Mutations</i> of membranes from <math>\Sigma_1^2</math> to <math>\Sigma_2^2</math> with different topologies.</li> <li>• <i>Loop order</i> <math>l</math> and the <i>Feynman diagram</i> of the process.</li> <li>• Relations between different equal-time slicings.</li> </ul>	<ul style="list-style-type: none"> <li>• <i>Morse function</i> <math>f</math> on <math>M^3</math>.</li> <li>• The family <math>f^{-1}(a)</math>, <math>a \in \mathbb{R}</math>.</li> <li>• Attaching of a 1-handle to <math>\Sigma_1^2 \cup \Sigma_2^2</math> with one end to <math>\Sigma_1^2</math> and the other to <math>\Sigma_2^2</math>.</li> <li>• Attaching of a 2-handle to <math>\Sigma^2</math> along a <i>separating</i> simple loop.</li> <li>• Attaching of 1-handles with both ends on <math>\Sigma_1^2</math> and/or 2-handles along a <i>non-separating</i> simple loop in <math>\Sigma_1^2</math>.</li> <li>• <i>Loop number</i> <math>l</math> of the graph <math>\Gamma</math> obtained by pinching every connected component of <math>f^{-1}(a)</math> into a point. Note that <math>l = 1 - \chi(\Gamma)</math>, where <math>\chi(\Gamma)</math> is the Euler characteristic of <math>\Gamma</math>.</li> <li>• Cerf's theory.</li> </ul>

(Cf. FIGURE 4.1.)

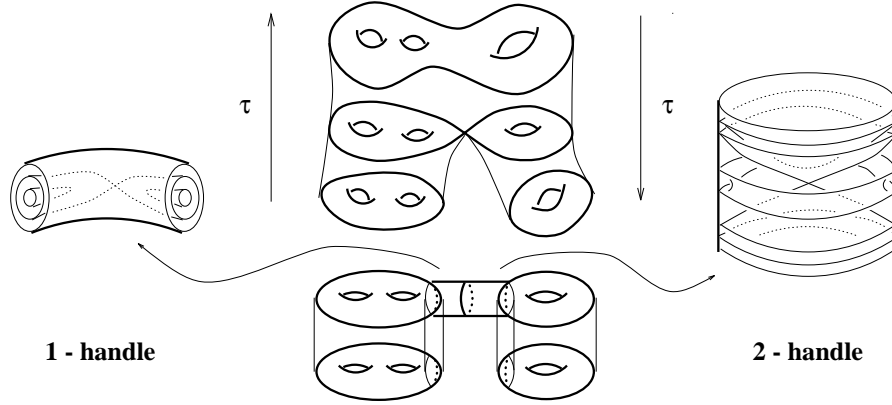


FIGURE 4.1. The relation among membrane interaction, equal-time slicing, and attaching of handles are indicated. Notice that, at 3-dimensions, attaching of a 1-handle turns into attaching of a 2-handle when the time direction is reversed.

This list describes well the Hamiltonian picture of string scatterings if one replaces "membrane" by "string", "1-" and "2-handle" at 3-dimensions by "1-handle" at 2-dimensions, etc., and recalls the Mandelstam diagrams for string world-sheets. However there are some new features that are not in common.

- (1) While strings do not have enough room for mutations, higher dimensional extended objects, e.g. membranes, do. In the latter case, due to (1, 2)-handle pair-creations and handle slidings, Feynman diagrams associated to a given  $M^3$  can be created, whose number of 2-valent vertices is greater than any given number.
- (2) While the loop order  $l$  thus described is well-defined for string world-sheets, it does depend on the equal-time slicing of the world-volume for higher dimensional extended objects, as indicated by the following example for membranes. In the latter case, it ranges from 0 (by handle-slidings so that all 1-handles come prior to 2-handles or equivalently by considering a Morse-Smale function) to the maximal rank of the free quotient of  $\pi_1(M^3)$ .<sup>2</sup> Hence, for example, every compact orientable  $M^3$  contributes to some tree-level scattering of membranes.

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<sup>2</sup> I would like to thank Prof. William Thurston for pointing this out to me and providing the example.

**Example 4.1.** Consider  $M^3 = S^2 \times S^1 - (B_-^3 \cup B_+^3)$ , which contributes to the membrane propagator from an incoming  $S_-^2 = \partial B_-^3$  to an outgoing  $S_+^2 = \partial B_+^3$ . The two different slicings of  $M^3$  as indicated in FIGURE 4.2 lead to Feynman diagrams of different loop orders. This is a general phenomenon and can be explained via handle slidings. In (a), only mutations of  $S^2$  are involved while, in (b), two Yukawa interactions are involved. Notice that  $\pi_1(S^2 \times S^1 - (B_-^3 \cup B_+^3)) = \mathbb{Z}$  has rank 1; hence this  $M^3$  contributes only to tree and 1-loop propagators from  $S_-^2$  to  $S_+^2$ .

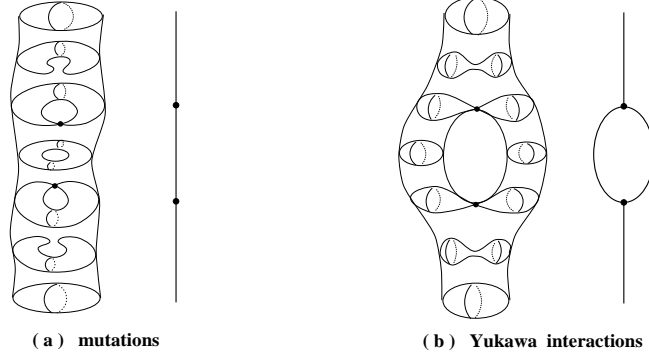


FIGURE 4.2. Two different slicings of  $S^2 \times S^1 - (B_-^3 \cup B_+^3)$  that lead to Feynman diagrams of different loop orders.

### The Lagrangian aspect.

When the membrane world-volume  $M^3$  is taken as a whole, two important elements for understanding its manifold structure are the *almost canonical decompositions* of  $M^3$ , following Kneser-Milnor-Waldhausen-Johannson-Jaco-Shalen (in historical order), and the *Thurston's geometrization program*, indicated in FIGURE 4.3 outlined from [Sco]. Though some last details remain conjectured or unpublished, the program depicts, among other things, a build-in complexity of a compact 3-manifold from its very own topology.

As already mentioned in [Su], there are a family of graphs associated to  $M^3$  following K-M-W-J-J-S-T, as illustrated in FIGURE 4.4 for  $M^3$  orientable. Complexity of  $M^3$ , as a membrane world-volume, can then be measured in terms of the loop number of these graphs and the complexity of the geometric pieces appearing as vertices of the graph. (For those pieces that are hyperbolic with finite volume, their hyperbolic volume may serve as a measure of complexity due to the following facts ([B-P], [Thu1]): (1) The set of volumes of (complete) hyperbolic 3-manifolds is well-ordered; and (2) the volume is a finite-to-one function of (complete) hyperbolic manifolds.)

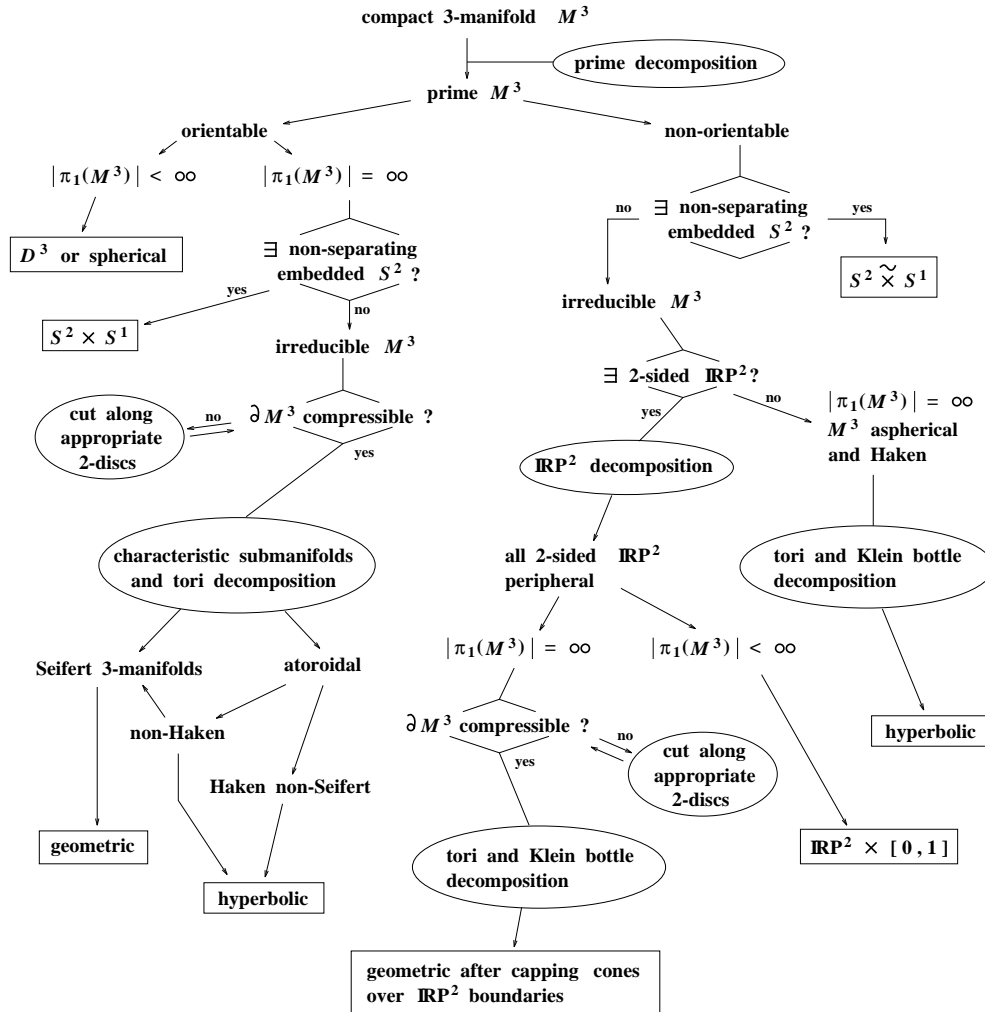
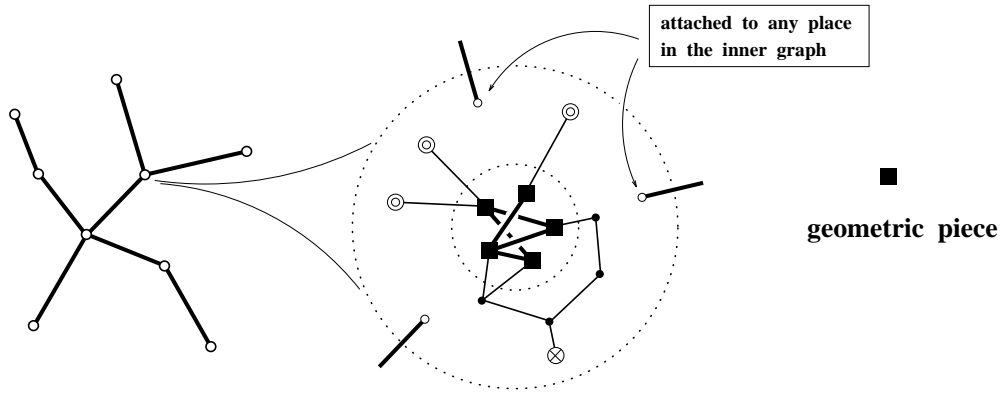


FIGURE 4.3. Decompositions and Thurston's geometrization program for compact 3-manifolds. The interior of the compact 3-manifold in every end of the flow chart admits a complete Riemannian structure locally modelled on one of the eight geometries:  $S^3$ ,  $\mathbb{E}^3$ ,  $\mathbb{H}^3$ ,  $S^2 \times \mathbb{E}$ ,  $\mathbb{H}^2 \times \mathbb{E}$ , Nil,  $SL(2; \mathbb{R})$ , and Sol. (One may call this "Thurston's eight-fold way".)



**Kneser - Milnor**

**Waldhausen - Johannson - Jaco - Shalen**

**Thurston**

FIGURE 4.4. An almost canonical graph associated to a compact orientable 3-manifold  $M^3$ , following K-M-W-J-J-S-T. In the figure,  $\odot$  indicates an incompressible boundary component;  $\otimes$  a compressible boundary component; the tree from a  $\otimes$  to a collection of  $\blacksquare$  indicates the chopping of a compressible boundary till it becomes incompressible - it corresponds to a co-dimension 0 submanifold-with-boundary in  $M^3$  with a decomposition by 2-handles. The propagator between  $\circ$  in the K-M part is  $S^2 \times [0, 1]$  while the propagator between  $\blacksquare$  in the W-J-J-S part is  $\mathbb{T}^2 \times [0, 1]$ .

On the other hand, instead of trying to fit  $M^3$  into a Feynman-diagram-like object, Milnor and Thurston explored the concept of *characteristic numbers* of 3-manifolds in [M-T], emphasizing the natural multiplicative property under covering maps. Up to an overall scaling, this characteristic number for hyperbolic 3-manifolds are their hyperbolic volumes. It is interesting to know that some connections between hyperbolic 3-manifolds and the physics of membranes have been pursued by Goncharov et al. (cf. [Go] and some sequels). The basic starting point there is the assumption that hyperbolic membrane world-volumes are the main contributors to the membrane partition function in the Minkowskian space-time. If it really turns out that these are the only 3-manifolds of physical significance in M-theory, then by coupling the volume-form of a hyperbolic  $M^3$  with a dilaton field - exactly like in string theory -, they can be related to some natural scales in M-theory (cf. [Sh1 - 2]).<sup>3</sup> In view of this, the right generalization of loop numbers for pointlike objects and strings to membranes may indeed be the Milnor-Thurston's characteristic numbers. If so, one would like to know *if there could be any Feynman diagram (and hence membrane scattering process) naturally associated to this number*.

*Remark 4.2.* Notice that there can be infinitely many (though only countable) topologically different  $M^3$  that share the same Betti numbers  $(b_1, b_2)$  (cf. Appendix). Thus they are not efficient in serving as the complexity or the loop number for  $M^3$ .

The two aspects of membrane scatterings and facts from 3-manifolds indicate that the nature of membrane particles can be very different from that of pointlike or stringy particles. We conclude with the hope that future collaborations of 3-dimensional geometers and string theorists can unveil this secret in M-theory.

## Appendix: A light exposition on homologies of 3-manifolds.

Since  $[M^3, N]$  depends quite on  $H_*(M^3; \mathbb{Z})$ , let us give a light exposition on the latter for completeness. We assume that  $M^3$  is compact orientable.

### General facts.

- (1)  $H_2(M^3; \mathbb{Z})$  is free abelian while  $H_1(M^3; \mathbb{Z})$ , being the abelianization of  $\pi_1(M^3)$ , can have torsion elements. Two basic tools for computing  $\pi_1(M^3)$

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<sup>3</sup> I would like to thank Prof. Emil Martinec for a discussion and Prof. Stephen Shenker for his talk at TASI-96, that draw my attention to this.



are the van Kampen's theorem and the homotopy sequence of a fibration ([Vi] and [Wh]).

(2) ([Mi2] and [Sta].)  $\pi_1(M_1^3 \sharp M_2^3) = \pi_1(M_1^3) * \pi_1(M_2^3)$  and

$$H_i(M_1^3 \sharp M_2^3; \mathbb{Z}) = H_i(M_1^3; \mathbb{Z}) \oplus H_i(M_2^3; \mathbb{Z}), \quad \text{for } i = 1, 2.$$

(3) For  $M^3$  closed,  $\chi(M^3) = 0$  - in fact  $T_*M^3$  is trivial ([Ki]) -. Hence  $b_1 = b_2$ . In general, by considering the double of the 3-manifold, one has  $b_2 - b_1 = \frac{1}{2}\chi(\partial M^3)$ .

### Examples.

**Example A.1.** *Homology 3-spheres:* ([Ro] and [Sti].) They have trivial homologies  $H_1(M^3; \mathbb{Z}) = H_2(M^3; \mathbb{Z}) = 0$  but possibly non-trivial  $\pi_1(M^3)$  ([Ro] and [Sti]). Notably the Poincaré's homology 3-sphere is the only known example that has finite  $\pi_1$ , of order 120.

**Example A.2.** *Lens space  $L(p, q)$ :* ([Ro] and [Sti].) They are constructed by identifying the boundary  $S^2$  of a 3-ball  $B^3$  by the map

$$(\theta, \phi) \longmapsto (-\theta, \phi + 2\pi \cdot \frac{q}{p}),$$

in terms of latitude and longitude.  $L(1, q) = S^3$  has  $H_1 = H_2 = 0$ .  $L(0, 1) = S^2 \times S^1$  has  $H_1 = H_2 = \mathbb{Z}$ . For all other  $L(p, q)$  with  $0 < q < p$  and  $p, q$  relatively prime,  $\pi_1 = \mathbb{Z}_p$  and hence  $H_1 = \mathbb{Z}_p$  and  $H_2 = 0$ .

**Example A.3.** *Knot complements in  $S^3$ :* ([Ro].) They all have  $H_1 = H_2 = \mathbb{Z}$  and  $\pi_i = 0$ , for  $i \geq 2$ .

The following two kinds of 3-manifolds are particularly akin to surfaces: *Seifert 3-manifolds* and *mapping tori* of a compact surface. In view of the aspect of membranes as excitations of strings, they may play special roles in M-theory.

**Example A.4.** *Seifert 3-manifolds:* They have been studied by Seifert [Se]. Their role in understanding the general structure of 3-manifolds is studied in [Ja], [Jo], and [J-S]; and their geometric structures are discussed in detail in [Sco].

In terms of the language of bundles, a Seifert 3-manifold  $M^3$  can be defined as the total space of an  $S^1$ -bundle  $\eta$  over a 2-orbifold  $Q$ . For  $M^3$  compact and

oriented, they are all irreducible except  $S^2 \times S^1$  and are determined by the following invariants:

$$e(\eta), \quad \chi(Q), \quad (p_i, q_i), \quad i = 1, \dots, r,$$

where  $e(\eta) \in \mathbb{Z}$  is the Euler number of the bundle,  $\chi(Q) \in \mathbb{Q}$  the Euler characteristic of the orbifold,  $r$  the number of exceptional fibers, and  $(p_i, q_i)$ ,  $p_i$  and  $q_i$  relatively prime, indicates that the exceptional fiber labelled  $i$  has a fibered solid torus neighborhood  $T(p_i, q_i)$  obtained by taking the oriented cylinder  $D^2 \times [0, 1]$ , foliated by  $[0, 1]$ , and gluing  $D^2 \times \{1\}$  to  $D^2 \times \{0\}$  with a twist by  $2\pi \frac{q_i}{p_i}$ .

For  $M^3$  closed oriented,  $\pi_1(M^3)$  has the following presentation [Se]:

(i) for  $Q$  orientable,

$$\begin{aligned} & \langle A_1, B_1, \dots, A_g, B_g, Q_0, Q_1, \dots, Q_r, H \mid \\ & A_i H A_i^{-1} = H^{\epsilon_i}, \quad B_i H B_i^{-1} = H^{\epsilon'_i}, \quad (i = 1, \dots, g; \epsilon_i, \epsilon'_i = \pm 1), \\ & Q_0 Q_1 \cdots Q_r = \prod_{i=1}^g [A_i, B_i], \\ & Q_j H Q_j^{-1} = H \quad (j = 0, 1, \dots, r), \\ & Q_0 H^b = Q_1^{p_1} H^{\hat{q}_1} = \cdots = Q_r^{p_r} H^{\hat{q}_r} = 1 \rangle; \end{aligned}$$

(ii) for  $Q$  non-orientable,

$$\begin{aligned} & \langle A_1, \dots, A_g, Q_0, Q_1, \dots, Q_r, H \mid \\ & A_i H A_i^{-1} = H^{\epsilon_i}, \quad (i = 1, \dots, g; \epsilon_i, \epsilon'_i = \pm 1), \\ & Q_0 Q_1 \cdots Q_r = \prod_{i=1}^g A_i^2, \\ & Q_j H Q_j^{-1} = H \quad (j = 0, 1, \dots, r), \\ & Q_0 H^b = Q_1^{p_1} H^{\hat{q}_1} = \cdots = Q_r^{p_r} H^{\hat{q}_r} = 1 \rangle, \end{aligned}$$

where  $g = 1 - \frac{1}{2}\chi(Q) - \frac{1}{2}\sum_{i=1}^r(1 - \frac{1}{p_i})$  for  $Q$  orientable,  $= 2 - \chi(Q) - \sum_{i=1}^r(1 - \frac{1}{p_i})$  for  $Q$  non-orientable, is the genus of  $Q$  (i.e. the number of handles for  $Q$  orientable or the number of crosscaps for  $Q$  non-orientable),  $b = e(\eta)$ , and  $0 < \hat{q}_i < p_i$  with  $q_i \hat{q}_i \equiv 1 \pmod{p_i}$ . Consequently,  $H_1(M^3; \mathbb{Z})$  is isomorphic to:

(i') For  $Q$  orientable,

$$\oplus_{2g} \mathbb{Z} \oplus \oplus_{r+2} \mathbb{Z} / \Gamma,$$

where  $\oplus_{2g} \mathbb{Z}$  is generated by  $A_i, B_i$ ,  $\oplus_{r+2} \mathbb{Z}$  by  $H, Q_j$ , and  $\Gamma$  is the lattice generated by

$$\begin{aligned} & \epsilon H, \quad \text{where } \epsilon = \max_i \{1 - \epsilon_i\} = 0 \text{ or } 2, \\ & Q_0 + Q_1 + \cdots + Q_r, \\ & Q_0 + b H, \quad p_1 Q_1 + \hat{q}_1 H, \quad \dots, \quad p_r Q_r + \hat{q}_r H. \end{aligned}$$

The rank of the above  $r + 3$  vectors is

$$\begin{aligned} r + 1 & \text{ if both } \epsilon \text{ and } bp_1 \dots p_r + \sum_{i=1}^{r-1} \hat{q}_i p_1 \dots \cancel{p_i} \dots p_{r-1} \text{ are 0, where} \\ & p_1 \dots \cancel{p_i} \dots p_{r-1} \text{ is the product } p_1 \dots p_{r-1} \text{ with } p_i \text{ deleted,} \\ r + 2 & \text{ otherwise.} \end{aligned}$$

(ii') For  $Q$  non-orientable,

$$\oplus_{g+r+2} \mathbb{Z} / \Gamma ,$$

where  $\oplus_{g+r+2} \mathbb{Z}$  is generated by  $A_i$ ,  $H$  and  $Q_j$ , and  $\Gamma$  is the lattice generated by

$$\begin{aligned} \epsilon H, \quad \text{where } \epsilon = \max_i \{1 - \epsilon_i\} = 0 \text{ or } 2, \\ Q_0 + Q_1 + \dots + Q_r - 2A_1 - \dots - 2A_g, \\ Q_0 + bH, \quad p_1 Q_1 + \hat{q}_1 H, \quad \dots, \quad p_r Q_r + \hat{q}_r H. \end{aligned}$$

The rank of the above  $r + 3$  vectors is  $r + 2$  if  $\epsilon = 0$ ,  $r + 3$  if  $\epsilon \neq 0$ .

**Example A.5.** *Mapping tori of compact surfaces:* ([C-B] and [Th5 – 6].) The mapping torus  $\Sigma_\phi$  of an automorphism  $\phi$  of a compact surface  $\Sigma$  is the 3-manifold obtained by first forming  $\Sigma \times [0, 1]$  and then gluing  $\Sigma \times \{1\}$  to  $\Sigma \times \{0\}$  via  $\phi$ . Naturally,  $\Sigma_\phi$  fibers over  $S^1$  with monodromy  $\phi$ . For  $\Sigma$  of negative Euler characteristic, the geometric structure of  $\Sigma_\phi$  is clarified by Thurston in [Th5], where it is shown that the interior of  $\Sigma_\phi$  either

- (i) admits a complete  $\mathbb{H}^2 \times \mathbb{E}$  structure of finite volume, and can be described as a Seifert fibration over some hyperbolic 2-orbifold,
- (ii) contains an embedded non-peripheral incompressible torus, which splits  $\Sigma_\phi$  into two simpler 3-manifolds, or
- (iii) (generic case) admits a complete hyperbolic structure of finite volume.

(Cases (i) and (ii) are not mutually exclusive, but (iii) excludes the other two.)

These three cases correspond exactly to his classification of surface automorphisms ([C-B] and [Th6]):

- (i')  $\phi$  is isotopic to an automorphism of finite order,
- (ii')  $\phi$  is isotopic to a reducible automorphism, which leaves invariant a system of simple loops, or
- (iii')  $\phi$  is pseudo-Anosov.

For homologies, let  $\gamma_1, \dots, \gamma_r$  be a set of generators for  $\pi_1(\Sigma, p_0)$  and  $\sigma$  be a generator for  $\pi_1(S^1)$ . Then

$$\pi_1(\Sigma_\phi) = \pi_1(\Sigma, p_0) * \pi_1(S^1) \Big/ \langle \sigma \gamma_i \sigma^{-1} (\phi_* \gamma_i)^{-1}; i = 1, \dots, r \rangle ,$$

where  $\langle \dots \rangle$  is the subgroup generated by  $\dots$ , and one identifies  $\pi_1(\Sigma, p_0)$  and  $\pi_1(\Sigma, \phi(p_0))$  by a fixed path connecting  $p_0$  and  $\phi(p_0)$ . After abelianization, one has

$$H_1(\Sigma_\phi; \mathbb{Z}) = H_1(\Sigma; \mathbb{Z}) \oplus H_1(S^1; \mathbb{Z}) \Big/ \langle \gamma_i - \phi_* \gamma_i; i = 1, \dots, r \rangle .$$

Finally, we conclude incompletely with the following broad and beautiful subject.

**Example A.6.** *Geometric 3-manifolds:* (E.g. [Sco] and [Th1 – 6].) Their fundamental groups are discrete subgroups of the group of isometries of the eight model geometries. The study of the case of hyperbolic 3-manifolds is majorly promoted by Thurston among other figures and is related to the study of Kleinian, Fuchsian, and quasi-Fuchsian groups. Nice expositions are given in, e.g. [Sco] and [Th1 – 3].

## References

- [As] P.S. Aspinwall, *An  $N = 2$  dual pair and a phase transition*, Nucl. Phys. **B460** (1996), pp. 57 – 76.
- [B-B-S] K. Becker, M. Becker, and A. Strominger, *Fivebranes, membranes and non-perturbative string theory*, Nucl. Phys. **B456** (1996), pp. 130 – 152.
- [B-P] R. Benedetti and C. Petronio, *Lectures on hyperbolic geometry*, Springer-Verlag, 1992.
- [B-T] R. Bott and L.W. Tu, *Differential forms in algebraic topology*, GTM 82, Springer-Verlag, 1982.
- [B-V-S] M. Bershadsky, C. Vafa, and V. Sadov, *D-strings on D-manifolds*, Nucl. Phys. **B463** (1996), pp. 398 – 414.
- [B- et al.] M. Berkooz, R.G. Leigh, J. Polchinski, J.H. Schwarz, N. Seiberg, and E. Witten, *Anomalies, dualities, and topology of  $D = 6$ ,  $N = 1$  superstring vacua*, Nucl. Phys. **B475** (1996), pp. 115 – 148.
- [C-B] A.J. Casson and S.A. Bleiler, *Automorphisms of surfaces after Nielsen and Thurston*, LMSST 9, Cambridge Univ. Press, 1988.
- [Ce] J. Cerf, *La stratification naturelle des espaces fonctions différentiables réelles et le théorème de la pseudo-isotopie*, Publ. Math. I.H.E.S. **39** (1970), pp. 5 – 173.

- [Co] S. Coleman, *Classical lumps and their quantum descendants* in *Aspects of symmetry*, pp. 185 – 264, Cambridge Univ. Press, 1985.
- [D-F-N] B.A. Dubrovin, A.T. Fomenko, and S.P. Novikov, *Modern geometry - methods and applications, Part II: The geometry and topology of manifolds*, GTM 104; *Part III: Introduction to homology theory*, GTM 124; Springer-Verlag, resp. 1984 and 1990.
- [D-K] S.K. Donaldson and P.B. Kronheimer, *The geometry of four-manifolds*, Oxford Univ. Press, 1990.
- [D-K-L] M.J. Duff, R.R. Khuri, and J.X. Lu, *String solitons*, *Phys. Reports* **259** (1995), pp. 213 – 326.
- [D-M] K. Dasgupta and S. Mukhi, *Orbifolds of M-theory*, *Nucl. Phys.* **B465** (1996), pp. 399 – 412.
- [F-N] P.G.O. Freund and R.I. Nepomechie, *Unified geometry of antisymmetric tensor gauge fields and gravity*, *Nucl. Phys.* **B 199** (1982), pp. 482 – 494.
- [Go] Yu.P. Goncharov, *Determinants of Laplacians in real line bundles over hyperbolic manifolds connected with quantum geometry of membranes*, *Lett. Math. Phys.* **19** (1990), pp. 73 – 81.
- [G-H] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons, Inc., 1978.
- [G-S-W] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring theory*, vol. 1 and vol. 2, Cambridge Univ. Press, 1987.
- [Ha] W. Haken, *Some results on surfaces in 3-manifolds*, in *Studies in modern topology*, P.J. Hilton ed., MAA Studies in Math., vol. 5, pp. 39 – 98, Math. Asso. Amer., 1968; printed by Prentice-Hall, Inc..
- [He] J. Hempel, *3-manifolds*, Ann. Math. Studies 86, Princeton Univ. Press, 1976.
- [Hi] P.J. Hilton, *On the homotopy groups of the union of spheres*, *J. London Math. Soc.* **30** (1955), pp. 154 – 172.
- [Hir] M. Hirsch, *The embedding of bounding manifolds in Euclidean space*, *Ann. Math.* **74** (1961), pp. 494 – 497.
- [Hu] S.-T. Hu, *Homotopy theory*, Pure Appl. Math. vol. 8, Academic Press, 1959.
- [HA-M-S] C. Hog-Angeloni, W. Metzler, and A.J. Sieradski eds., *Two-dimensional homotopy and combinatorial group theory*, London Math. Soc. Lect. Notes Ser. 197, Cambridge Univ. Press, 1993.
- [H-W1] P. Hořava and E. Witten, *Heterotic and type I string dynamics from eleven dimensions*, *Nucl. Phys.* **B460** (1996), pp. 506 – 524.
- [H-W2] ———, *Eleven-dimensional supergravity on a manifold with boundary*, *Nucl. Phys.* **B475** (1996), pp. 94 – 114.
- [Ja] W. Jaco, *Lectures on three-manifold topology*, Regional Conf. Ser. Math. no. 43, Amer. Math. Soc., 1980.
- [Jo] K. Johannson, *Homotopy equivalence of 3-manifolds with boundaries*, Lect. Notes Math. no. 761, Springer-Verlag, 1979.

- [J-S] W. Jaco and P.B. Shalen, *A new decomposition theorem for irreducible sufficiently-large 3-manifolds*, in *Algebraic and geometric topology, Part 2*, R.J. Milgram ed., Proc. Symp. Pure Math. vol. 32, pp. 71 – 84, Amer. Math. Soc., 1978.
- [Ki] R.C. Kirby, *The topology of 4-manifolds*, Lect. Notes Math. no. 1374, Springer-Verlag, 1989.
- [K-M] D. Kutasov and E.J. Martinec, *New principles for string/membrane unification*, [hep-th/9602049](#).
- [K-M-O'L] D. Kutasov, E.J. Martinec, and M. O'Loughlin, *Vacua of M-theory and  $N = 2$  strings*, [hep-th/9603116](#).
- [L-M] H. Luckock and I. Moss, *The quantum geometry of random surfaces and spinning membranes*, *Class. Quantum Grav.* **6** (1989), pp. 1993 – 2027.
- [Ma] E.J. Martinec, *Geometric structures of M-theory*, [hep-th/9608017](#).
- [Mi1] J. Milnor, *On simply connected 4-manifolds*, *Symposium Internacional de Topología Algebraica*, La Univ. Nacional Autónoma de México, UNESCO, 1958.
- [Mi2] ———, *A unique factorization theorem for 3-manifolds*, *Amer. J. Math.* **84** (1962), pp. 1 – 7.
- [Mi3] ———, *Topology from the differentiable viewpoint*, Univ. Press of Virginia, 1965.
- [M-B] J.W. Morgan and H. Bass eds., *The Smith conjecture*, Pure Appl. Math. vol. 112, Academic Press, 1984.
- [M-T] J. Milnor and W.P. Thurston, *Characteristic numbers of 3-manifolds*, *L'Enseignement Math.* **23** (1977), pp. 249 – 254.
- [Ne1] R.I. Nepomechie, *Approaches to a non-abelian antisymmetric tensor gauge field theory*, *Nucl. Phys.* **B212** (1983), pp. 301 – 320.
- [Ne2] ———, *Magnetic monopoles from antisymmetric tensor gauge fields*, *Phys. Rev.* **D 31** (1985), pp. 1921 – 1924.
- [Po] V. Poénaru, *On the geometry of differentiable manifolds*, in *Studies in modern topology*, P.J. Hilton ed., MAA Studies in Math. vol. 5, pp. 165 – 207, Math. Asso. Amer., 1968; printed by Prentice-Hall, Inc..
- [P-C-J] J. Polchinski, S. Chaudhuri, and C.V. Johnson, *Notes on D-branes*, [hep-th/9602052](#).
- [Ro] D. Rolfsen, *Knots and links*, Math. Lect. Ser. 7, 2nd printing, Publish or Perish, Inc., 1990.
- [Sch] C. Schmidhuber, *D-brane actions*, *Nucl. Phys.* **B467** (1996), pp. 146 – 158.
- [Sco] P. Scott, *The geometry of 3-manifolds*, *Bull. London Math. Soc.* **15** (1983), pp. 401 – 487.
- [Sh1] S.H. Shenker, *The strength of non-perturbative effects in string theory*, in *Random surfaces and quantum gravity*, O. Alvarez, E. Marinari, and P. Windey eds., pp. 191 – 200, Plenum Press, 1991.
- [Sh2] ———, *Another length scale in string theory?*, [hep-th/9509132](#).
- [Se] H. Seifert, *Topologie dreidimensionaler gefaserte Räume*, *Acta Math.* **60** (1933), pp. 147 – 238. (English translation by W. Heil in *A textbook of topology*, H. Seifert and W. Threlfall, Pure and Appl. Math. 89, Academic Press, 1980.)

- [Si] J. Singer, *Three-dimensional manifolds and their Heegaard diagrams*, Trans. Amer. Math. Soc. **35** (1933), pp. 88 – 111.
- [Sp1] E.H. Spanier, *The homology of Kummer manifolds*, Proc. Amer. Math. Soc. **7** (1956), pp. 155 – 160.
- [Sp2] ———, *Algebraic topology*, Springer-Verlag, 1966.
- [Sta] J. Stallings, *Group theory and three-dimensional manifolds*, Yale Math. Monographs 4, Yale Univ. Press, 1971.
- [Sti] J. Stillwell, *Classical topology and combinatorial group theory*, 2nd ed., GTM 72, Springer-Verlag, 1993.
- [Su] A. Sugamoto, *Starting from strings to membranes*, in *Supermembranes and physics in 2 + 1 dimensions*, M.J. Duff, C.N. Pope, and E. Sezgin eds., pp. 16 – 28, World Scientific, 1990.
- [Th1] W.P. Thurston, *The geometry and topology of three-manifolds*, Princeton Lecture Notes, 1979. Also revised version 1990.
- [Th2] ———, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982), pp. 357 – 381.
- [Th3] ———, *Hyperbolic geometry and 3-manifolds*, in *Low-dimensional topology*, R. Brown and T.L. Thickstun eds., LMS 48, pp. 9 – 25, Cambridge Univ. Press, 1982.
- [Th4] ———, *Hyperbolic structures on 3-manifolds, I: Deformation of acylindrical manifolds*, Ann. Math. **124** (1986), pp. 203 – 246.
- [Th5] ———, *Hyperbolic structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle*, Princeton preprint, 1986.
- [Th6] ———, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. **19** (1988), pp. 417 – 431.
- [Vi] J.W. Vick, *Homology theory: an introduction to algebraic topology*, Pure Appl. Math. vol. 53, Academic Press, 1973. Reprinted, 2nd ed., GTM 145, Springer-Verlag, 1994.
- [Wa] F. Waldhausen, *Recent results on sufficiently large 3-manifolds*, in *Algebraic and geometric topology, Part 2*, R.J. Milgram ed., Proc. Symp. Pure Math. vol. 32, pp. 21 – 38, Amer. Math. Soc., 1978.
- [Wh] G.W. Whitehead, *Elements of homotopy theory*, GTM 61, Springer-Verlag, 1978.
- [Whi] J.H.C. Whitehead, *On simply-connected 4-dimensional polyhedra*, Comment. Math. Helv. **22** (1949), pp. 48 – 92.
- [Wi1] E. Witten, *String theory dynamics in various dimensions*, Nucl. Phys. **B 443** (1995), pp. 85 – 126.
- [Wi2] ———, *Five-branes and M-theory on an orbifold*, Nucl. Phys. **B463** (1996), pp. 383 – 397.
- [Wi3] ———, *Phase transitions in M-theory and F-theory*, Nucl. Phys. **B471** (1996), pp. 195 – 216.
- [Y-Z] S.-T. Yau and E. Zaslow, *BPS states, string duality, and nodal curves on  $K3$* , Nucl. Phys. **B471** (1996), pp. 503 – 512.